A note on a model to evaluate acquisition price and quantity of used products for remanufacturing

April 13, 2015

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Abstract

Pokharel and Liang [2012. A model to evaluate acquisition price and quantity of used products for remanufacturing. International Journal of Production Economics 138, 170–176] considered a consolidation center that buys used products of different quality levels and sells them together with spare parts to a remanufacturer. The consolidation center's decision problem is to determine the acquisition price to offer for used products and the quantities of spare parts to buy. In this paper, comments on their work are given. It is shown that following Pokharel and Liang's original assumptions, the problem has a trivial solution. We then consider an alternative assumption where supply is uniform and depends on the acquisition price. For this setting, an efficient solution algorithm and numerical examples are provided. In a second model, additional assumptions are relaxed, allowing the consolidation center more flexibility. As expected, this further decreases cost.

Keywords: Reverse logistics, Remanufacturing, Collection of used products, Pricing of used products

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1 Introduction

In recent years, remanufacturing has become increasingly popular for ecological as well as economic reasons. The remanufacturing process starts with the reclamation of used products, often called “cores”. They are then disassembled, cleaned and inspected. Depending on the quality of the cores, some spare parts may be added and, finally, they are reassembled to some sort of “as good as new” products. In this context, Pokharel and Liang (2012) consider a consolidation center that buys used products from collection centers (which obtained them from customers), combines them with appropriate spare parts corresponding to their quality level and sells both to a remanufacturer. Given a fixed order quantity from the remanufacturer that must be fulfilled and stochastic returns of used products, Pokharel and Liang (2012) propose a model to determine optimal acquisition prices and quantities for the different quality levels. More specifically, they do not decide on the quantity of used products actually bought, but on the planned quantity that equals the number of corresponding spare parts that must be bought in advance before the realization of supply. Moreover, the planned quantities (total number of spare parts) must sum up to the given order size. For reasons of business continuity, everything offered by the collection centers is actually bought.

This paper is organized as follows. In Section 2, comments on the work of Pokharel and Liang (2012) are given. We strictly adhere to Pokharel and Liang’s assumptions and identify several shortcomings of their paper. To do so, in Subsection 2.1, we carve out a main assumption that is not explicitly stated in Pokharel and Liang (2012): Despite being a decision variable in their model, the acquisition price does not influence supply. The amount and quality of cores obtained by the consolidation center is independent of the acquisition price. Thus, the only cost-minimizing solution is obviously the lowest acquisition price possible. However, Pokharel and Liang (2012) do not obtain this trivial solution because of a sign error in their analysis of the KKT-conditions, as we show in Subsection 2.2. From our point of view, the existence of the trivial solution renders any further analysis of the problem as given by Pokharel and Liang’s assumptions superfluous. For the sake of completeness, we discuss in Subsection 2.3 why the numerical solution procedure developed and used in the remainder of Pokharel and Liang (2012) is highly questionable and does not even get close to the optimal solution in the instances considered.

In Section 3, we present our first model. It is obtained by correcting the key assumption. We now assume that supply depends on the acquisition price offered by the consolidation center. Albeit also other assumptions could be questioned, we think this is the smallest change necessary to arrive at a reasonable problem. Moreover, it ensures analytical tractability. To improve readability, we state the complete problem formulation in Subsection 3.1 and also briefly motivate our choice of the price-dependent supply function. In Subsection 3.2, we derive the KKT conditions from the corresponding optimization problem and show that the trivial solution is now no longer necessarily optimal. A solution algorithm is developed in Subsection 3.3 and applied to numerical examples in Subsection 3.4.

In Section 4, a second model is presented. Here, we additionally relax three questionable assumptions from Pokharel and Liang (2012). First, the consolidation center is no longer required to buy all cores offered to him. Second, we now assume that the quality levels are nested in the sense that the spare parts necessary for a low-quality core are also sufficient for a higher-quality core. Third, the total number of spare parts bought is no longer required to equal the given order size, for example allowing the consolidation center to buy more spare parts to hedge against supply uncertainty.

We conclude in Section 4.

2 Comments

2.1 Dependence of used product supply on acquisition price

Pokharel and Liang (2012) never explicitly state how the supply of used products depends on the acquisition price. Their model assumptions only state that "used product supply at quality level n, S_n, is stochastic […]" (P&L Assumption 1), "Used product supply quantity at quality level n follows a probability density function $f(S_n)$ with known mean μ_n and standard deviation σ_n" (P&L Assumption 2) and that the acquisition price must be in the range between the salvage value r_0 and the per unit underage penalty cost P_0 minus the cost of the corresponding spare parts b_n (P&L Assumption 5: $r_0 < p_n < P_0 - b_n$).

Comment. By comparing equations (P&L 5) and (P&L 6) we note that the derivative of $S_n p_n$ with respect to p_n is obviously S_n (see P&L 6). There is no dependence of the returned quantity S_n on the acquisition price p_n: S_n is not a function of p_n. Given that the acquired quantity of used products does not depend on the acquisition price, one would intuitively expect the lowest possible price to minimize cost. More formally, the objective function stated in Section 2.2 is linear in the prices p_n and increasing. As it is minimized, the smallest possible values are optimal. However, in the two remarks in their Section 3.3, Pokharel and Liang analytically show that the optimal price p_n does not equal the lower or upper bound. They consider this result intuitive because they seem to be not aware of the fact that their model technically does not include any influence of prices on supply.

Moreover, from equations (P&L 3) and (P&L 4) and later elaborations, it is obvious that the inequality in P&L Assumption 5 is not meant in the strict sense, that is, it should read $r_0 \le p_n \le P_0 - b_n$.

2.2 Analysis of the KKT conditions

In the following, we first briefly restate the authors' analytical investigation. Then, comments are given. It is shown that the authors' counterintuitive result is caused mainly by a sign error when applying KKT-conditions.

The starting point for their elaborations is "the cost function, C, for total acquisition [cost] by the consolidation center" (Pokharel and Liang 2012, Section 3.3)

$$C\left(p_n, q_n\right) = \sum_{n=1}^{K}\left[S_n p_n + b_n q_n + P_0 \int_0^{q_n} \left(q_n - S_n\right) f\left(S_n\right) dS_n - r_0 \int_{q_n}^{\infty} \left(S_n - q_n\right) f\left(S_n\right) dS_n\right] \quad \text{(P\&L 1)}$$

where the first three elements are costs for acquisition of used products, spare parts and underage quantities, respectively, and the fourth is the salvage value obtained from an overage quantity. To obtain the optimal acquisition price p_n and planned acquisition quantity q_n for each quality level n, (P&L 1) is minimized subject to the following constraints:

$$\sum_{n=1}^{K} q_n = d \quad \text{(P\&L 2)}$$

$$p_n \ge r_0 \qquad n = 1,\ldots,K \quad \text{(P\&L 3)}$$

$$p_n \le P_0 - b_n \qquad n = 1,\ldots,K \quad \text{(P\&L 4)}$$

These constraints ensure that the sum of the planned acquisition quantities over all quality levels equals the order quantity from the remanufacturer and that the acquisition price is in the range mentioned above. Using (P&L 1–4), the authors derive the Lagrangian

$$\begin{aligned} L = &\sum_{n=1}^{K}\left[S_n p_n + b_n q_n + P_0 \int_0^{q_n} \left(q_n - S_n\right) f\left(S_n\right) dS_n - r_0 \int_{q_n}^{\infty} \left(S_n - q_n\right) f\left(S_n\right) dS_n\right] \\ &-\lambda\left(\sum_{n=1}^{K} q_n - d\right) - \alpha_n\left(r_0 - p_n\right) - \beta_n\left(p_n + b_n - P_0\right) \end{aligned} \quad \text{(P\&L 5)}$$

where λ, α_n, and β_n are the Lagrange multipliers associated with the total quantity of used products as well as the lower and upper bounds on the acquisition price, respectively. From the Lagrangian, the following KKT first order conditions are derived using $F(q_n)$ to denote the cumulative probability density function of S_n:

$$\frac{\partial L}{\partial p_n} = S_n + \alpha_n - \beta_n = 0 \tag{P\&L 6}$$

$$\frac{\partial L}{\partial q_n} = b_n + P_0 F(q_n) + r_0(1 - F(q_n)) - \lambda = 0 \tag{P\&L 7}$$

$$\sum_{n=1}^{K} q_n = d\,,\ \lambda \geq 0\,,\ \lambda\left(\sum_{n=1}^{K} q_n - d\right) = 0 \tag{P\&L 8}$$

$$r_0 - p_n \leq 0\,,\ \alpha_n \geq 0\,,\ \alpha_n(r_0 - p_n) = 0 \tag{P\&L 9}$$

$$p_n + b_n - P_0 \leq 0\,,\ \beta_n \geq 0\,,\ \beta_n(p_n + b_n - P_0) = 0 \tag{P\&L 10}$$

Using these conditions, the authors now show in two remarks that optimal prices p_n cannot be equal to the lower bound, but may be equal to the upper bound.

- "If $p_n = r_0$ and $p_n < P_0 - b_n$, then $\alpha_n \geq 0$ [(P&L 9)] and $\beta_n = 0$ [… (P&L 10)]. Otherwise, it will give $S_n \leq 0$ or strictly $S_n = 0$ [… (P&L 6)]". They conclude that offering the lowest price is not optimal.
- "If, $p_n > r_0$ and $p_n = P_0 - b_n$ then $\alpha_n = 0$ [… (P&L 9)] and $\beta_n \geq 0$ [… by (P&L 10)]. Then from [… (P&L 6)], $S_n = \beta_n$, […]." The authors conclude that offering the highest possible price can be optimal. This is described as intuitive because "such a high price can attract the return of more used products".

Comment. Given the stochastic environment and the last two terms, equation (P&L 1) is obviously meant to represent expected cost. Thus, to be formally precise, the first element should be $\mathbf{E}[S_n p_n] = \mu_n p_n$. Correcting for obvious typos such as the omitted last sum, the Lagrangian is given by

$$\begin{aligned} L(\mathbf{p},\mathbf{q},\boldsymbol{\alpha},\boldsymbol{\beta},\lambda) = & \sum_{n=1}^{K}\left[\mu_n p_n + b_n q_n + P_0\int_0^{q_n}(q_n - S_n)\,f(S_n)\,dS_n - r_0\int_{q_n}^{\infty}(S_n - q_n)\,f(S_n)\,dS_n\right] \\ & -\lambda\left(\sum_{n=1}^{K} q_n - d\right) - \sum_{n=1}^{K}\left[\alpha_n(r_0 - p_n) + \beta_n(p_n + b_n - P_0)\right] \end{aligned} \tag{P\&L 5new}$$

with $\mathbf{p} = (p_1,\dots,p_K)$ and $\mathbf{q} = (q_1,\dots,q_K)$ as well as the Lagrange multipliers $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_K)$, $\boldsymbol{\beta} = (\beta_1,\dots,\beta_K)$ and λ associated with the lower and upper bounds on the acquisition price and the total quantity of used products, respectively. From (P&L 5new), the following KKT necessary conditions are obtained (see e.g. Taha 2007, Chapter 18.2.2):

$$\frac{\partial L}{\partial p_n} = \mu_n + \alpha_n - \beta_n = 0 \qquad n = 1,\dots,K \tag{P\&L 6new}$$

$$\frac{\partial L}{\partial q_n} = b_n + P_0 F(q_n) + r_0 (1 - F(q_n)) - \lambda = 0 \qquad n = 1,...,K \qquad \text{(P\&L 7new)}$$

$$\sum_{n=1}^{K} q_n = d\,,\ \lambda \in \mathbb{R}\,,\ \lambda\left(\sum_{n=1}^{K} q_n - d\right) = 0 \qquad \text{(P\&L 8new)}$$

$$r_0 - p_n \le 0\,,\ \alpha_n \le 0\,,\ \alpha_n (r_0 - p_n) = 0 \qquad n = 1,...,K \qquad \text{(P\&L 9new)}$$

$$p_n + b_n - P_0 \le 0\,,\ \beta_n \le 0\,,\ \beta_n (p_n + b_n - P_0) = 0 \qquad n = 1,...,K \qquad \text{(P\&L 10new)}$$

The main difference to conditions (P&L 6–10) is the sign of the Lagrange multipliers: Conditions (P&L 9new) and (P&L 10new) now require α_n and β_n to be non-positive. This is obviously caused by accounting for the minimization and the inequality constraints. Using the corrected conditions, we obtain exactly the opposite results regarding prices equaling the upper and lower bound (assuming the bounds are not equal). Optimal prices now cannot equal the upper bound or any value between the bounds with a positive acquisition quantity, but can equal the lower bound.

- If $p_n = P_0 - b_n$ it follows that $p_n > r_0$ and, thus, $\alpha_n = 0$ (P&L 9new) and $\beta_n \le 0$ (P&L 10new). From (P&L 6new), $\mu_n \le 0$ is obtained, which is a contradiction for reasonable values of μ_n. Thus, offering prices equal to their upper bound is not optimal.
- Moreover, if $p_n > r_0$ and $p_n < P_0 - b_n$, it follows that $\alpha_n = 0$ (P&L 9new) and $\beta_n = 0$ (P&L 10new). From (P&L 6new), $\mu_n = 0$ is obtained. Thus, offering prices strictly greater than the lower bound cannot be optimal.
- If $p_n = r_0$ for some quality level n, it follows that $p_n < P_0 - b_n$ and, thus, $\alpha_n \le 0$ (P&L 9new) and $\beta_n = 0$ (P&L 10new). From (P&L 6new), $\mu_n \ge 0$ is obtained. Thus, only offering the lowest prices is optimal. Thus, we analytically showed that the optimal prices equal the lower bound and confirmed the trivial and intuitive solution.

The authors do not discuss sufficiency of the KKT conditions. With reasonable assumptions ($\mu_n \ge 0\ \forall n$ and $r_0 < P_0$, see P&L Assumption 5) it is easy to see that the objective function is convex. As all constraints are linear, the necessary KKT conditions are also sufficient.

2.3 Solution heuristic

After the two remarks, the authors essentially state that the optimal acquisition quantity q_n is obtained from (P&L 7) and (P&L 8) and an additional equation that reflects that "the critical probabilities for such optimal quantity balance at a point where the expected profit is equal to the expected loss for all used products":

$$F(q_n)(P_n - p_n) = (1 - F(q_n))(p_n - r_0) \qquad \text{(P\&L 16)}$$

The authors state that the left hand side is “the unit expected profit by selling one more used product” and the right hand side is “the unit expected loss of one more unsold used product.”

Subsequently, they present an iterative algorithm that searches for q_n that simultaneously satisfy (P&L 7) and (P&L 8). Finally, p_n are calculated using (P&L 16).

Comment. Given the existence of the trivial solution shown in the preceding section, any further analysis as done in the remainder of Pokharel and Liang (2012) – the second part of P&L Section 3.3, Section 3.4, and Section 4 – is actually superfluous. However, we wondered why the solution algorithm presented in Section 3.3 does not even get close to the trivial optimal solution in Section 4.

Regarding (P&L 16), note that balancing expected loss and additional profits associated with increasing a decision variable is broadly applied. For example, this concept is the basis for the well-known Littlewood’s rule (see Littlewood 1972) that has heavily influenced revenue management. However, it remains unclear what the LHS and RHS of (P&L 16) exactly represent and the explanations signal that not all relevant values are considered.

As a used product supplied and bought by the consolidation center can only be sold to the manufacturer if an adequate spare part was bought, the acquisition quantity q_n and, thus, the number of spare parts to buy should be the relevant decision variable here. The two cases to distinguish are now as follows:

- An additional spare part (the q_n-th) is bought and a cost of b_n for the spare part is incurred. With probability $F(q_n-1)$, it is not used because supply is insufficient. In this case, an underage cost of P_0 is incurred. With probability $1-F(q_n-1)$, supply is sufficient, p_n is paid and no underage cost is incurred (for the q_n-th core).
- The q_n-th spare part is not bought. In this case, the underage cost P_0 is always incurred, but with probability $1-F(q_n-1)$, supply is at least q_n, p_n is paid and the q_n-th core can be salvaged for r_0.

Putting this together, we obtain $b_n + F(q_n-1)\cdot P_0 + \left(1-F(q_n-1)\right)\cdot p_n =$ $P_0 + \left(1-F(q_n-1)\right)\cdot(p_n - r_0)$ which simplifies to

$$b_n = \left(1-F(q_n-1)\right)\cdot(P_0 - r_0). \qquad \text{(P\&L 16new)}$$

This equation is now very similar to Littlewood’s rule and can be readily interpreted. Buying an additional spare part costs b_n and, if supply is high enough, saves $P_0 - r_0$ because the consolidation center does not have to pay the underage cost but at the same time cannot obtain the salvage value. From (P&L 16new), it becomes clear that the optimal quantities do not depend on the acquisition price, if $F(q_n)$ is independent of the price. Thus, (P&L 16new) can be transparently derived and has a clear interpretation, whereas no justification for (P&L 16) can be found.

3 A first model with uniform supply distribution

In this section, we present a model that closely follows the assumptions of Pokharel and Liang (2012) but assumes that although supply is stochastic, the collection centers tend to collect more used products if the acquisition price offered to them is higher. In Section 4, we will additionally change other questionable assumptions. To improve readability, the following Subsection 3.1 briefly states all assumptions, including those identical to Pokharel and Liang (2012). In Subsection 3.2, the model formulation is given and sufficient conditions for an optimal solution are derived. In Subsection 3.3, an iterative algorithm to solve the model is presented. Subsection 3.4 presents a numerical example based on the data given by Pokharel and Liang (2012). The notation used throughout this section is summarized in Table 1.

3.1 Problem statement

In this subsection, we completely restate the problem to improve readability. This enables us to concisely define the relevant parameters. We also motivate our deviation in one key assumption from Pokharel and Liang (2012), namely the dependence of supply on acquisition price.

We consider a consolidation center that has already contractually agreed to supply d units of cores to a remanufacturer. Thus, for the decision problem considered here, the price that the consolidation center obtains from the remanufacturer is irrelevant and d is given. There are K quality levels, and cores of each quality level $n \in \{1,...,K\}$ are shipped together with specific spare parts to the remanufacturer. The cores are bought at an acquisition price p_n from collection centers that provide a stochastic supply denoted by the random variable S_n. Before knowing the realization of supply, the consolidation center has to partition the remanufacturer's total order size d into the planned acquisition quantities q_n for each quality level and buys the required spare parts at a per unit price of b_n. If the collection center cannot provide the required quantity d because supply falls short of q_n for some quality levels, a per unit penalty cost of P_0 is incurred. This cost P_0 depends on the contract with the remanufacturer. It may just capture lost sales and equal the selling price, but it can additionally include a contractual penalty or reflect a loss of goodwill. On the other hand, if supply S_n exceeds q_n, the consolidation center nonetheless buys the whole amount provided to facilitate the business of the collection centers and disposes of the superfluous quantity $S_n - q_n$ at a salvage value of r_0. It is assumed that if the acquisition price was below r_0, the collection centers would salvage the used products themselves. Thus, the acquisition price p_n for quality level n must obviously satisfy $r_0 \le p_n \le P_0 - b_n$. Only a single period without any inventories, technology improvements, etc. is considered.

Table 1: Notation and indices

$n = 1, ..., K$: quality level	S_n : used product supply (random variable)
p_n : acquisition price	$f_{n,p_n}(S_n)$: pdf of S_n
q_n : planned acquisition quantity	l_n : scaling parameter
b_n : cost of required spare parts	μ_n : expected value of S_n
d : fixed order size of the remanufacturer	σ_n : standard deviation of S_n
P_0 : penalty payment for shortages, cost for lost sales (per unit)	$\lambda, \boldsymbol{\alpha}, \boldsymbol{\beta}$: Lagrange multipliers
r_0 : salvage value of surplus units of used products	$L(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \lambda)$: Lagrange function
$C(\mathbf{p}, \mathbf{q})$: total cost of the consolidation center	

In addition to these assumptions from Pokharel and Liang (2012), we now assume that stochastic supply S_n depends on the acquisition price p_n. With higher acquisition prices, the collection centers increase their effort and tend to collect more used products.

That acquisition prices can be used to control returns is not only intuitive, but also a widespread assumption in the current literature (see Gönsch 2014 and the references cited therein). For example, Bulmus et al. (2014) use it to analyze competition for used products and Cai et al. (2014) to study acquisition and production planning for a joint manufacturing and remanufacturing system. Xiong et al. (2014) investigate dynamic pricing for used products with lost sales and uncertain quality. Gönsch (2014) considers negotiations to acquire used products and contains references to various prior studies and surveys. Although these studies do not consider the unique multi-stage setting with collection and consolidation centers of Pokharel and Liang (2012), we think that this basic economic principle can also be applied here. With a higher acquisition price, the collection centers increase their efforts, or, in line with the abovementioned literature, simply offer a higher price to the customers, leading to an increase in the supply of used products.

Analogously to Gönsch (2014) and the references cited therein, we assume that customers' valuations for their used products are heterogeneous and the amount of used products returned to the collection centers is stochastic. Moreover, as discussed above, we assume that a higher acquisition price leads to an increase in supply. Accordingly, and to improve tractability, we assume that supply for the consolidation center is zero if the price equals the salvage value and is uniformly distributed in the interval $\left[0, l_n\left(p_n - r_0\right)\right]$ for $p_n > r_0$, where l_n is a scaling parameter. The corresponding pdf (for $p_n > r_0$) is then given by

$$f_{n,p_n}\left(S_n\right) = \begin{cases} \dfrac{1}{l_n\left(p_n - r_0\right)} & \mathit{if}\ 0 \le S_n \le l_n\left(p_n - r_0\right) \\ 0 & \mathit{otherwise} \end{cases} \tag{1}$$

The consolidation center's decision problem now is to determine the optimal acquisition prices $\mathbf{p} = \left(p_1, ..., p_K\right)$ and planned acquisition quantities $\mathbf{q} = \left(q_1, ..., q_K\right)$ that minimize expected cost given the fixed order size d, the cost parameters $\mathbf{b} = \left(b_1, ..., b_K\right)$, r_0, and P_0 as well as the scaling parameters $\mathbf{l} = \left(l_1, ..., l_K\right)$.

3.2 Model formulation

Using this notation, the decision problem is formally given by

$$\begin{aligned} &\min C\left(\mathbf{p}, \mathbf{q}\right) \\ &= \sum_{n=1}^{K}\left[\frac{p_n l_n}{2}\left(p_n - r_0\right) + b_n q_n + P_0 \int_0^{q_n}\left(q_n - S_n\right) f_{n,p_n}\left(S_n\right) dS_n - r_0 \int_{q_n}^{l\left(p_n - r_0\right)}\left(S_n - q_n\right) f_{n,p_n}\left(S_n\right) dS_n\right] \end{aligned} \tag{2}$$

subject to

$$\sum_{n=1}^{K} q_n = d \tag{3}$$

$$p_n \ge r_0 \qquad n = 1, ..., K \tag{4}$$

$$p_n \le P_0 - b_n \qquad n = 1, ..., K. \tag{5}$$

The objective function (2) calculates expected cost and is a sum over all quality levels. For each quality level n, the first term represents the expected payments to the collection centers, the second term is the cost for spare parts, the third term is the penalty cost incurred when the supplied quantity is too low and the fourth term refers to the revenue from salvaging surplus supply. Constraint (3) ensures that the planned quantities q_n are a partition of the order size d. Constraints (4) and (5) reflect the lower and upper bound for the acquisition prices p_n, respectively.

Remark. To be formally precise, we must use limits to account for the pdf used being valid only for $p_n > r_0$. As this would become superfluous again through rearrangements in Subsection 3.3, we neglect this to improve readability. Moreover, we need to include a constraint ensuring non-negativity of q_n. However, as the unconstrained problem's solution has nonnegative values for q_n for realistic parameter values, we neglect this constraint to render the presentation clearer. For completeness, a model including this constraint is briefly considered in Appendix A.

For model (2) – (5), the Lagrangian is given by (see e.g. Taha 2007, Chapter 18.2.2)

$$L(\mathbf{p},\mathbf{q},\boldsymbol{\alpha},\boldsymbol{\beta},\lambda)$$
$$=\sum_{n=1}^{K}\left[\frac{p_n l_n}{2}(p_n-r_0)+b_n q_n+P_0\int_0^{q_n}(q_n-S_n)\,f_{n,p_n}(S_n)\,dS_n-r_0\int_{q_n}^{l(p_n-r_0)}(S_n-q_n)\,f_{n,p_n}(S_n)\,dS_n\right] \quad (6)$$
$$-\lambda\left(\sum_{n=1}^{K}q_n-d\right)-\sum_{n=1}^{K}\left[\alpha_n(r_0-p_n)+\beta_n(p_n+b_n-P_0)\right]$$

with $\mathbf{p}=(p_1,...,p_K)$ and $\mathbf{q}=(q_1,...,q_K)$ as well as the Lagrange multipliers $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_K)$, $\boldsymbol{\beta}=(\beta_1,...,\beta_K)$ and λ associated with the lower and upper bound constraints on the acquisition price and the total quantity of used products, respectively. From (6), the following KKT necessary conditions are obtained:

$$\frac{\partial L}{\partial p_n}=l_n(p_n-r_0)-\frac{q_n^2(P_0-r_0)}{2l_n(p_n-r_0)^2}+\alpha-\beta=0 \qquad n=1,...,K \qquad (7)$$

$$\frac{\partial L}{\partial q_n}=b_n+\frac{q_n(P_0-r_0)}{l_n(p_n-r_0)}+r_0-\lambda=0 \qquad n=1,...,K \qquad (8)$$

$$\sum_{n=1}^{K}q_n=d\,,\ \lambda\in\mathbb{R}\,,\ \lambda\left(\sum_{n=1}^{K}q_n-d\right)=0 \qquad (9)$$

$$r_0-p_n\le 0\,,\ \alpha_n\le 0\,,\ \alpha_n(r_0-p_n)=0 \qquad n=1,...,K \qquad (10)$$

$$p_n+b_n-P_0\le 0\,,\ \beta_n\le 0\,,\ \beta_n(p_n+b_n-P_0)=0 \qquad n=1,...,K \qquad (11)$$

In Appendix B, we show that these necessary conditions are also sufficient. Now, the simple analysis from Section 2.2 showing that the optimal price is equal the lower bound is no longer possible.

3.3 Solution algorithm

Unfortunately, Equations (7)–(11) are difficult to solve analytically. Therefore, we present an iterative algorithm here. Please note that the algorithm itself is completely different from Pokharel and Liang's, although the overall structure shows some similarities at first glance.

We now solve equations (7) and (8) simultaneously for p_n and q_n. First, we neglect (10) and (11) and denote the resulting price by p'_n:

$$p'_n=r_0+\frac{(\lambda-b_n-r_0)^2}{2(P_0-r_0)} \qquad n=1,...,K \qquad (12‘)$$

Considering (10) and (11), we obtain

$$p_n = \frac{\beta-\alpha}{l_n} + r_0 + \frac{\left(\lambda - b_n - r_0\right)^2}{2\left(P_0 - r_0\right)} \qquad n = 1,\dots,K \tag{12}$$

Comparing (12') and (12) it is obviously necessary to choose the Lagrange multipliers as follows to take Equations (10) and (11) into account:

$$\alpha_n = \min\left\{0, l_n\left(p_n' - r_0\right)\right\} \qquad n = 1,\dots,K \tag{13}$$

and

$$\beta_n = \min\left\{0, l_n\left(P_0 - b_n - p_n'\right)\right\} \qquad n = 1,\dots,K \tag{14}$$

For q_n, we obtain

$$q_n = \frac{\left(\lambda - b_n - r_0\right) l_n\left(p_n - r_0\right)}{P_0 - r_0} \qquad n = 1,\dots,K. \tag{15}$$

As p_n is nondecreasing in λ and q_n increases in λ and p_n (for reasonable values of λ, otherwise see Appendix A for $q_n < 0$), both p_n and q_n are nondecreasing in λ. Thus, it is easy to find a λ such that Equation (9) is holds. We only briefly outline the procedure.

Algorithm 1: Determining the solution of (7)–(11)

1. *Start with an arbitrary value* $\lambda \geq \max_n\left\{b_n + r_0\right\}$.
2. *Calculate all* p_n' *using (12').*
3. *Calculate all* α_n *and* β_n *using (13) and (14).*
4. *Calculate all* p_n *using (12).*
5. *Calculate all* q_n *using (15).*
6. *If* $\sum_{n=1}^{K} q_n \approx d$ *: break*
   elseif $\sum_{n=1}^{K} q_n < d$ *: increase* λ *and go to Step 2*
   else: decrease λ *and go to Step 2*

3.4 Numerical example

We illustrate the algorithm with three examples based on data given by Pokharel and Liang (2012), assuming a product with $K = 6$ quality levels. The salvage value is $r_0 = \$10$ and the penalty payment is $P_0 = \$100$. The data for $\mathbf{b} = \left(b_1,\dots,b_6\right)$ and $\mathbf{l} = \left(l_1,\dots,l_6\right)$ is given together with the resulting minimum value for λ (see Step 1 of the algorithm) in Table 2. For the equality check in Step 6, a precision of 0.01 is used.

Table 2: Data used for numerical study

Quality level n	Cost of spare parts b_n [$]	Scaling parameter l_n [Units/$]	$b_n + r_0$
1	10	54	20
2	15	42	25
3	20	58	30
4	25	116	35
5	30	100	40
6	35	353	45

In the first scenario, the consolidation center is contractually obliged to deliver $d = 2{,}000$ units. The optimal solution is at $\lambda^* = 72.019$ with a total cost of $124,090. Data related to the 6 quality levels given in Table 3. As the prices are not equal to their lower or upper bounds, $\boldsymbol{\alpha}^* = \boldsymbol{\beta}^* = \mathbf{0}$.

Table 3: Prices p_n and planned quantities q_n with resulting distribution of supply (first scenario: $d = 2{,}000$)

Quality level n	Price p_n [$]	Quantity q_n [Units]	Expected value μ_n	Standard deviation σ_n
1	25.03	469.21	405.90	8.22
2	22.28	269.50	257.92	6.56
3	19.81	265.61	284.46	6.89
4	17.61	363.26	441.57	8.58
5	15.70	202.63	284.78	6.89
6	14.06	429.80	715.83	10.92

To further illustrate the model, we additionally consider a second scenario with a smaller agreed order size of only $d = 1{,}000$ units. This scenario reflects the idea behind the second scenario considered by Pokharel and Liang (2012), but the change of the order size allows us to save space and reuse all other data previously given. The optimal solution now is at $\lambda^* = 64.126$ with a cost of $55,697. Data related to the 6 quality levels is given in Table 4 and again $\boldsymbol{\alpha}^* = \boldsymbol{\beta}^* = \mathbf{0}$.

Table 4: Prices p_n and planned quantities q_n with resulting distribution of supply (second scenario: $d = 1{,}000$)

Quality level n	Price p_n [$]	Quantity q_n [Units]	Expected value μ_n	Standard deviation σ_n
1	20.82	286.39	292.06	6.98
2	18.50	155.28	178.60	5.46
3	16.47	142.28	187.62	5.59
4	14.71	176.92	273.34	6.75
5	13.23	86.68	161.68	5.19
6	12.03	152.45	358.68	7.73

An analysis of the cost components (not given here) shows for the two examples that roughly 30% is due to underage costs, and that underage costs are often only slightly higher than costs for spare parts. Moreover, most of the time when underage costs occur, the bottleneck is not the overall supply quantity, but a mismatch between cores and spare parts in some quality levels. Thus, the consolidation center can probably reduce cost by buying more spare parts (although some might definitely remain unused) or by – alternatively – using the same spare parts for different quality levels. We investigate these issues in the following section.

4 A second model with relaxed assumptions

In this section, we additionally relax three questionable assumptions from Pokharel and Liang (2012). In Subsection 4.1, the corresponding model formulation is given. In Subsection 4.2, we revisit the examples from Section 3.4 and show that with the relaxed assumptions, the firm can operate at a considerably lower cost. We reuse most of the notation from Section 3, however, some new notation is necessary and summarized in Table 5.

4.1 Model formulation

The assumptions changed are as follows. First, the consolidation center is no longer required to buy all cores offered. Second, we now assume that the quality levels are nested in the sense that the spare parts necessary for a low-quality core are also sufficient for a higher-quality core. In line with the numerical examples in Section 3, we define that lower indices n denote higher quality. Third, the total number of spare parts bought is no longer required to equal the given order size, for example allowing the consolidation center to buy more spare parts to hedge against supply uncertainty.

Table 5: Additional notation introduced in Section 4

t_n : quantity of spare parts bought	$C(\mathbf{p},\mathbf{t})$: total cost of the consolidation center
$Q_n(\mathbf{S},\mathbf{t},d)$: quantity of cores acquired	

In our second model, the consolidation center decides on the acquisition prices p_n and the number of spare parts to buy t_n. We use the new notation t_n here to distinguish it from the planned acquisition quantity q_n of Section 3. The quantity of cores acquired $Q_n(\mathbf{S},\mathbf{t},d)$ is formally no decision variable. Instead, this second stage decision is directly calculated by

$$Q_n(\mathbf{S},\mathbf{t},d) = min\left\{S_n, t_n + \sum_{n'=n+1}^{K} t_{n'} - Q_{n'}(\mathbf{S},\mathbf{t},d), d - \sum_{n'=n+1}^{K} Q_{n'}(\mathbf{S},\mathbf{t},d)\right\} \qquad n = 1,...,K \quad (16)$$

Equation (16) is based on the nesting of the quality levels. Thus, a high-quality core is at least as valuable as a lower-quality core to the consolidation center, because if there are spare parts for the lower-quality one, then they can also be used for the high-quality core. It follows that the consolidation center will never offer more for a lower quality core, that is $p_n \geq p_{n'}$ for $n \leq n'$ (for reasonable supply functions). Having this in mind, the consolidation center obviously buys as much low-quality cores as possible, given a realization of supply $\mathbf{S}$, available spare parts $\mathbf{t}$ and the order size d. The number of cores $Q_K(\mathbf{S}, \mathbf{t}, d)$ acquired in the lowest-quality level, K, can be directly computed by (16) as the sums contain no elements and, thus, there is no recursion. It is the minimum of three terms: the number of cores supplied S_K, the number of spare parts available t_K, and the order size d. Regarding a higher quality level n, the structure remains the same. The number of cores acquired cannot exceed supply S_n, spare parts available and demand. However, regarding the spare parts available, spare parts bought for lower quality levels $n' > n$ but still unused have to be added. Similarly, the order size is reduced by the number of (cheaper) cores already acquired.

Using $Q_n(\mathbf{S}, \mathbf{t}, d)$, the decision problem is formally given by

$$
\begin{aligned}
&\min C(\mathbf{p}, \mathbf{t}) \\
&= \int_0^{1_1(p_1 - r_0)} \cdots \int_0^{1_K(p_K - r_0)} \left(\sum_{n=1}^{K} \left[Q_n(\mathbf{S}, \mathbf{t}, d)\, p_n + b_n t_n \right] + P_0 \left(d - \sum_{n=1}^{K} Q_n(\mathbf{S}, \mathbf{t}, d) \right) \right) \prod_{n=1}^{K} f_{n, p_n}(S_n) d\mathbf{S}
\end{aligned}
\tag{17}
$$

subject to

$$p_n \geq r_0 \qquad n = 1, \ldots, K \tag{18}$$

$$t_n \geq 0 \qquad n = 1, \ldots, K. \tag{19}$$

The objective function (17) calculates expected cost. For a given realization of supply $\mathbf{S}$, the first sum represents payments to the collection centers and cost for spare parts. The second term is the penalty cost incurred when number of cores bought and combined with spare parts falls short of the order size. Constraint (18) is a lower bound on p_n necessary for our supply distribution and constraint (19) ensures nonnegativity of t_n.

Compared to model (2)–(5), model (17)–(19) is unfortunately less analytically tractable because the quality levels cannot be considered independently. As is obvious from the definition of $Q_n(\mathbf{S}, \mathbf{t}, d)$, the number of spare parts of quality level n to buy depends on all lower quality levels $n' > n$. Thus, the integral over the joint distribution is needed to calculate the expectation.

4.2 Numerical example

In the following, we reconsider the two examples from Section 3.4 and solve them with model (17)–(19). As no analytical solution is possible, we use a numerical approach. The model was implemented in Matlab R2014b, the integral numerically approximated and the minimization was performed by the metaheuristic patternsearch from the Global Optimization Toolbox.

In the first scenario, the consolidation center is contractually obliged to deliver $d = 2{,}000$ units. The solution now has a total cost of only $99,302. Data related to the 6 quality levels given in Table 6.

Compared to the more restrictive model of Section 3, higher prices are offered and more spare parts for lower quality cores (levels 3 to 6) are bought. Both is intuitive, higher prices increase supply and hurt less as no longer all cores supplied must be bought. Buying relatively more spare parts for lower quality cores makes the consolidation center more flexible, as they can also be used for higher quality cores. Interestingly, at least with the given prices, the consolidation center still buys exactly 2,000 spare parts. Apparently, it is better to substitute a cheap spare part with a more versatile one than to buy additional cheap ones. Exploiting the additional flexibility, the consolidation center decreases cost by 20%. Now, only about 7% of the cost is for underage (compared to 30% in Section 3) and about 52% and 41% is for spare parts and the acquisition of cores, respectively.

Table 6: Prices p_n and spare parts bought t_n with resulting distribution of supply (first scenario: $d = 2{,}000$)

Quality level n	Price p_n [$]	Spare parts t_n [Units]
1	32.20	102.25
2	26.64	270.00
3	22.05	283.75
4	19.47	413.00
5	19.99	302.00
6	14.25	629.00

In the second scenario the agreed order size is only $d = 1{,}000$ units. The optimal solution now has a cost of $43,653, which is 22% less than before. Data related to the 6 quality levels is given in Table 7.

Compared to Section 3, the consolidation center again offers higher prices and buys more spare parts for lower quality cores (levels 3 to 6) to hedge against supply uncertainty. Total cost consists of about 6% for underage, 54% for spare parts and 40% for cores. Again, the total number of cores bought equals the order size.

Table 7: Prices p_n and spare parts bought t_n with resulting distribution of supply (second scenario: $d = 1{,}000$)

Quality level n	Price p_n [$]	Spare parts t_n [Units]
1	24.80	88.97
2	20.69	155.03
3	20.33	192.00
4	16.46	301.00
5	13.49	111.00
6	11.84	152.00

5 Conclusion

In this paper, we identified several shortcomings in Pokharel and Liang (2012) and corrected them in two ways. First, sticking to their original assumptions, it was shown that the problem only has a trivial solution. The solution algorithm given is highly questionable, because it has no justification and it fails to identify this solution. We then reasonably modified a key assumption and allowed supply to depend on the acquisition price. For this first model, a new solution algorithm was given and illustrated using numerical examples. Of course, several of the assumptions we kept could also be challenged. Thus, we presented a second model without these assumptions. In this model, the consolidation center can freely choose the number of spare parts bought, buys only as many cores as needed, and can use spare parts for lower quality cores for high quality cores instead. In our examples, these additional flexibilities reduce cost by about 20%.

Future work could consider further relaxing the remaining assumptions. For example, the nesting structure for the relation between cores and spare parts of different quality levels is still restrictive. Non-ordered quality levels (e.g. either part A, or B, or both of a core may be broken) could be considered. However, this might necessitate solving a linear program instead of the recursive formulation used in this work and, thus, further complicate the calculation of the number of cores acquired. Another possible extension is to internalize the decisions on d and the selling price of the consolidation center to the remanufacturer.

Appendix

A. Ensuring non-negativity of q

In this appendix, we consider the case that the problem unconstrained with regard to q_n may have an optimal solution with $q_n < 0$. Therefore, we include the constraint

$$q_n \geq 0 \qquad n = 1,...,K. \qquad \text{(A.1)}$$

For model (2)–(5) with (A.1), the Lagrangian is given by

$$\begin{aligned} & L(\mathbf{p},\mathbf{q},\boldsymbol{\alpha},\boldsymbol{\beta},\lambda) \\ & = \sum_{n=1}^{K}\left[\frac{p_n l_n}{2}(p_n - r_0) + b_n q_n + P_0 \int_0^{q_n}(q_n - S_n)\, f_{n,p_n}(S_n)\, dS_n - r_0 \int_{q_n}^{l(p_n - r_0)}(S_n - q_n)\, f_{n,p_n}(S_n)\, dS_n\right] \\ & -\lambda\left(\sum_{n=1}^{K} q_n - d\right) - \sum_{n=1}^{K}\left[\alpha_n(r_0 - p_n) + \beta_n(p_n + b_n - P_0) - \gamma_n q_n\right] \end{aligned} \qquad \text{(A.2)}$$

with the new Lagrange multipliers $\boldsymbol{\gamma} = (\gamma_1,...,\gamma_K)$ associated with the new lower bound (A.1) for q_n. Using (A.2), the second KKT necessary condition (8) slightly changes and we obtain new necessary conditions:

$$\frac{\partial L}{\partial q_n} = b_n + \frac{q_n(P_0 - r_0)}{l_n(p_n - r_0)} + r_0 - \lambda + \gamma_n = 0 \qquad n = 1,...,K \qquad \text{(A.3)}$$

$$-q_n \leq 0,\ \gamma_n \leq 0,\ \gamma_n q_n = 0 \qquad n = 1,...,K. \qquad \text{(A.4)}$$

With condition (A.1), the simultaneous solutions of equations (7) and (8) for p_n and q_n now depend on γ_n:

$$p_n = \frac{\beta - \alpha}{l_n} + r_0 - \frac{(\lambda - \gamma_n - b_n - r_0)^2}{2(P_0 - r_0)} \qquad n = 1,...,K \qquad \text{(A.5)}$$

$$q_n = \frac{(\lambda - \gamma_n - b_n - r_0)\, l_n (p_n - r_0)}{P_0 - r_0} \qquad n = 1,...,K. \qquad \text{(A.6)}$$

Given (A.5) and (A.6), a simple binary search as in Algorithm 1 is no longer possible and a solution regarding λ and $\boldsymbol{\gamma}$ has to be found simultaneously. Note that the new necessary conditions (7), (9)–(11), (A.3), and (A.4) are again sufficient because the Hessian matrix of the Lagrangian is still given by (B.3) (see Appendix B) and the new constraint (A.1) is linear.

B. Sufficiency of KKT necessary conditions (7)–(11)

As it is easy to see that the constraints (3)–(5) are linear, it suffices to show that the objective function (2) is convex to proof sufficiency of the necessary conditions for a minimization problem. We notice that (2) can be written as

$$C(\mathbf{p},\mathbf{q}) = \sum_{n=1}^{K} C_n(\mathbf{p},\mathbf{q}) \tag{B.1}$$

with

$$C_n(\mathbf{p},\mathbf{q}) = \frac{p_n l_n}{2}(p_n - r_0) + b_n q_n + P_0 \int_0^{q_n} (q_n - S_n) f_{n,p_n}(S_n)\, dS_n - r_0 \int_{q_n}^{l(p_n - r_0)} (S_n - q_n) f_{n,p_n}(S_n)\, dS_n \,. \tag{B.2}$$

As the sum of convex functions is convex, it suffices to show that (B.2) is convex. The Hessian matrix of (B.2) is given by

$$H(C_n(\mathbf{p},\mathbf{q})) = \begin{bmatrix} \dfrac{\partial^2 C}{\partial^2 p_n} & \dfrac{\partial^2 C}{\partial p_n \partial q_n} \\ \dfrac{\partial^2 C}{\partial q_n \partial p_n} & \dfrac{\partial^2 C}{\partial^2 q_n} \end{bmatrix} = \begin{bmatrix} l_n + \dfrac{q_n^2 (P_0 - r_0)}{l_n (p_n - r_0)^3} & \dfrac{-q_n (P_0 - r_0)}{l_n (p_n - r_0)^2} \\ \dfrac{-q_n (P_0 - r_0)}{l_n (p_n - r_0)^2} & \dfrac{P_0 - r_0}{l_n (p_n - r_0)} \end{bmatrix}. \tag{B.3}$$

We now apply Sylvester's criterion to show that $H(C_n(\mathbf{p},\mathbf{q}))$ is positive definite because its leading principal minors are all positive. Regarding the first leading principal minor, we obtain

$$\det H^1(C_n(\mathbf{p},\mathbf{q})) = l_n + \frac{q_n^2 (P_0 - r_0)}{l_n^2 (p_n - r_0)^3} > 0\,. \tag{B.4}$$

Regarding the second leading principal minor, we obtain

$$\det H^2(C_n(\mathbf{p},\mathbf{q})) = det \begin{bmatrix} l_n + \dfrac{q_n^2 (P_0 - r_0)}{l_n (p_n - r_0)^3} & \dfrac{-q_n (P_0 - r_0)}{l_n (p_n - r_0)^2} \\ \dfrac{-q_n (P_0 - r_0)}{l_n (p_n - r_0)^2} & \dfrac{P_0 - r_0}{l_n (p_n - r_0)} \end{bmatrix} = \frac{P_0 - r_0}{p_n - r_0} > 0\,. \tag{B.5}$$

Both inequalities are obtained using (3) and (4) and hold for all inner solutions of problem (2)–(5). Thus, $C_n(\mathbf{p},\mathbf{q})$ is convex and $C(\mathbf{p},\mathbf{q})$ is also convex.

□

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