Optimizing Conditional Value-at-Risk in Dynamic Pricing

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Abstract

Many industries use dynamic pricing on an operational level to maximize revenue from selling a fixed capacity over a finite horizon. Classical risk-neutral approaches do not accommodate the risk aversion often encountered in practice. We add to the scarce literature on risk aversion by considering the risk measure Conditional Value-at-Risk (CVaR), which recently became popular in areas like finance, energy or supply chain management. A key aspect of this paper is selling a single unit of capacity, which is highly relevant in, for example, the real estate market. We analytically derive the optimal policy and obtain structural results. The most important managerial implication is that the risk-averse optimal price is constant over large parts of the selling horizon, whereas the price continuously declines in the standard setting of risk-neutral dynamic pricing. This offers a completely new explanation for the price-setting behavior often observed in practice. For arbitrary capacity, we develop two algorithms to efficiently compute the value function and evaluate them in a numerical study. Our results show that applying a risk-averse policy, even a static one, often yields a higher CVaR than applying a dynamic, but risk-neutral, policy.

Keywords: Revenue Management, Dynamic Pricing, Dynamic Programming, Risk Management, Service Operations

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1 Introduction

Dynamic pricing considers the problem of selling a fixed capacity of a perishable product during a given selling horizon. Demand is stochastic and the aim is to "dynamically" adjust the price during the selling horizon based on remaining time and capacity, such that the total expected revenue is maximized. In practice, dynamic pricing was first applied in, for example, the airline, hotel, and car rental industries (Talluri and van Ryzin 2004). Today, it is increasingly used for many more products, including, for example, professional sports and concert tickets (e.g., Rovell 2011, Economist 2011). Dynamic pricing is also relevant for the real estate industry (e.g., Berkovec and Goodman 1996, Meng et al. 2007, Besbes and Maglaras 2012).

In the first industries using dynamic pricing, the high repetition of events justified risk neutrality. Moreover, the resulting maximization of the expected revenue led to more tractable models. However, the assumption of risk neutrality is not always appropriate (Feng and Xiao 1999). There are industries in which the selling process is rarely repeated. A large percentage of (private) sellers who only have one house for sale characterizes the market for private homes. Another example is an event promoter who organizes very few, but large, concerts per year (Levin et al. 2008). She therefore invests a great deal of her capital to present each event, and it is rational that she should lower her expectations about revenue to safeguard against adverse situations, for example, poor ticket sales, because each concert is a matter of economic existence. Furthermore, a risk-neutral model might be insufficient if a steady stream of revenue has to be ensured to, for example, match the financial liabilities (Lancaster 2003), or please shareholders. In addition to the issues raised above regarding the appropriateness of risk neutrality in certain settings, the literature (e.g., Bitran and Caldentey 2003, Weatherford 2004), as well as our own consulting experience, suggests that, in practice, decision makers tend to be generally risk averse.

Recently, coherent risk measures (Artzner et al. 1999) became a popular tool to capture risk aversion in various areas (e.g. finance) because of their desirable properties. These integrated measures comprise both the mean and variability measures. Four axioms (convexity, monotonicity, translation equivariance, and positive homogeneity) are imposed to guarantee consistency with intuition about rational risk-averse decision making. The most popular coherent risk measure is the Conditional Value-at-Risk (CVaR). For continuous distributions, CVaR is simply the expectation below a given quantile, the Value-at-Risk (VaR) at a certain probability level α.

In practice, a main feature of CVaR is the fact that risk is measured in monetary units. Its popularity and the simple definition make the risk measure and its probability level α easy to communicate to senior management or, more generally, to people with a scarce background in probability (see, e.g., Luciano et al. 2003 or Koenig and Meissner 2015b). Thus, CVaR is comparatively easy to interpret. For a level of $\alpha = 0$, only the worst case is considered, whereas for $\alpha = 1$, the expected value is considered. By contrast, utility functions are hard for practitioners to understand (Gotoh and Takano 2007) and difficult to interpret. For example, the exponential utility function widely used in academia has one parameter whose actual value is almost impossible to interpret besides the fact that risk-aversion decreases with the parameter's value. However, both CVaR and utility functions are difficult to calibrate. Although there are some approaches to calibrate utility functions, they are not really applicable in practice (see, e.g., Xu 2010 or Choi and Ruszczyński 2008). Likewise, we are not aware of any method to measure a decision-maker's probability level α for CVaR. In academia, a given risk-aversion is assumed. In practice, CVaR is often calibrated by imposing a maximum cost of risk-aversion. For example, in the Basel II framework VaR's α was chosen to obtain "approaches which are both more comprehensive and more sensitive to risks than the 1988 Accord, while maintaining the overall level of regulatory capital." (Bank for International Settlements 2001, p. 2).

Accordingly, coherent risk measures and especially CVaR are increasingly used to consider risk aversion in operations management models. For example, Gotoh and Takano (2007) show that it is usually very hard to obtain analytical solutions when using CVaR in a newsvendor problem. Subsequently, inventory and newsvendor problems are optimized by several authors including Ahmed et al. (2007), Choi and Ruszczyński (2008), Chen et al. (2009), Xu (2010), Choi et al. (2011), as well as Chen et al. (2015). CVaR is also applied in the risk-averse management of energy sources and/or storage (see, e.g., García-González et al. 2007, Pousinho et al. 2012, and González et al. 2014).

This paper makes the following contributions: We consider dynamic pricing with a risk-averse seller maximizing CVaR over the selling horizon (see Section 3.4 for a formal problem statement). Therefore, we present a dynamic model by means of the Bellman equation of the corresponding value function. We then reformulate the value function, which leads to a convex function. For one unit of capacity, this allows us to analytically derive and analyze the CVaR-optimal policy with some mild assumptions. The most interesting property is that a fixed price policy is optimal in the beginning of the selling horizon, whereas the price declines from the beginning in the standard risk-neutral setting (see, e.g., Talluri and van Ryzin 2004, (Chapter 5)). In terms of arbitrary capacity, we develop two efficient algorithms for computing the value function. An important result of our extensive numerical study is that applying a risk-averse policy, even a static one, is far better than applying a dynamic, but risk-neutral, policy.

Please note that the maximization of CVaR of total revenue (that is, over the entire booking horizon) comes at a price. There is no meaningful interpretation of the decision problems considered during the booking horizon, other than that they are consistent with this goal (Pflug and Pichler 2016). However, we think this is only a slight caveat in the context of dynamic pricing, where the time horizon considered is generally rather short (see, e.g., Gallego and van Ryzin 1994 or Barz 2007, p. 95) and only total revenue and its distribution are considered in practice and literature.

The structure of this paper is as follows: Section 2 provides a review of the relevant literature. Section 3 presents the model we use to optimize the CVaR in a dynamic pricing environment. Section 4 presents the structural properties of the optimal policy for the special case of one unit of capacity, as well as the corresponding analytical formulation of the value function. Section 5 presents two algorithms that numerically solve the problem efficiently for arbitrary capacities. In Section 6, we conduct an extensive numerical study, showing the efficiency of our newly developed algorithms, and comparing them with several benchmark mechanisms. Finally, we conclude with a discussion of our results in Section 7. An online supplement contains the proofs.

2 Literature review

In the following, we briefly discuss the literature most relevant to this research. For an extensive overview on revenue management with risk-aversion we refer to Gönsch (2017).

2.1 Risk-averse dynamic pricing

Authors who studied inter-temporal price discrimination (e.g., Stokey 1979, Landsberger and Meilijson 1985, and Wilson 1988) about 30 years ago laid the basis of dynamic pricing. Research on dynamic pricing gained momentum with the seminal paper by Gallego and van Ryzin (1994), who considered optimal dynamic pricing of a single product with stochastic demand over a finite selling horizon. In the two decades since then, hundreds of follow-up papers have been published. Several review articles (e.g., Bitran and Caldentey 2003, Chiang et al. 2007, and, with a special focus, den Boer 2015 and Gönsch et al. 2013) and textbooks (e.g. Talluri and van Ryzin 2004 (Chapter 5) and Phillips 2005 (Chapter 10)) have structured and summarized this research.

Feng and Xiao (1999) were the first to introduce risk aversion in a dynamic pricing framework. They considered a model with a pair of pre-determined prices, but instead of maximizing just the expected revenue, used an objective function that takes business risk into account by adding a penalty for revenue variance. In addition to the primary objective of optimizing the expected revenue, Levin et al. (2008) also require a fixed minimum revenue with at least a given probability. A drawback of this target criterion is that it neglects the distribution of revenue below the minimum revenue, whereas our approach takes the complete tail distribution into account. Another group of authors captures risk attitudes via utility functions. Lim and Shanthikumar (2007) show the equivalence of risk-averse, single-product dynamic pricing with an exponential utility function and robust dynamic pricing which takes demand model errors into account (see Wang and Xiao 2017 for a recent paper from that stream). Li and Zhuang (2009) show that the well-known monotonicities from risk-neutral dynamic pricing are preserved under risk aversion with additive general utility and atemporal exponential utility functions. Moreover, these authors show that the optimal price decreases in risk aversion. Schlosser (2015, 2016) consider atemporal exponential utility functions. Schlosser (2015) endogenizes the decision about the advertising intensity. He is the first to derive optimal closed-form solutions for risk-averse dynamic pricing, with the exception of the highly restricted setting of Feng and Xiao (1999). Schlosser (2016) considers multi-product dynamic pricing. The products have independent demands and inventories, but are related through adoption effects. That is, sales of one product can depend on past sales of all products. Simulations show that the variance can be significantly reduced, while expected profits are still near optimal. Koenig and Meissner (2010) use standard deviation and (Conditional) Value-at-Risk to evaluate dynamic pricing policies. To the best of our knowledge, the optimization of CVaR in the context of dynamic pricing has not yet been considered in the literature, although substantial arguments (see above) speak in its favor.

2.2 Risk-averse capacity control

In capacity control, the firm influences demand by controlling the availability of predefined products with fixed prices (see, e.g., the textbooks by Talluri and van Ryzin 2004 (Chapters 2 and 3) and Phillips 2005 (Chapters 7 and 8)) for problems with risk-neutral decision makers). Weatherford (2004) modified the famous EMSR-b heuristic by substituting revenues with a risk-averse utility function. Barz (2007), Barz and Waldmann (2007), and Feng and Xiao (2008) use an exponential utility function, but they work with the original DP formulation. Zhuang and Li (2011) examine optimal booking limits with an atemporal utility function. Koenig and Meissner (2015b, 2016) consider target percentile risk and VaR. Gönsch and Hassler (2014) develop a heuristic to optimize CVaR. Huang and Chang (2011) and Koenig and Meissner (2015a) modify existing approaches by heuristically incorporating risk aversion parameters and propose hands-on formulae to calculate them. Koch et al. (2016) demonstrate that this approach can be used to tailor every control approach to arbitrary risk measures and that the parameters can be obtained from simulation based optimization.

3 Modeling CVaR in dynamic pricing

In the following, we introduce the setting with its notation (Section 3.1) and restate the Bellman equation for the classical risk-neutral case (Section 3.2). Based on the formal introduction of CVaR in Section 3.3, we formally state the problem of maximizing CVaR in dynamic pricing (Section 3.4) and develop a recursive formulation that optimizes the CVaR by means of the Bellman equation (Section 3.5).

3.1 Setting and notation

Gallego and van Ryzin (1994) introduced the standard dynamic pricing setting. In a market with imperfect competition, a firm influences demand through price variations. The firm has a given stock of C units and replenishment is not possible. The stock can only be sold during a finite selling horizon and any remaining units are worthless. The selling horizon is discretized into T periods, which are indexed backwards in time, i.e., periods T and 0 mark the beginning and the end of the selling horizon. In each period t, and with remaining capacity c, the firm decides on the selling price $r_{t,c}$. The willingness-to-pay (WTP) of the customer arriving in period t is an i.i.d. continuous random variable X_t with cumulative distribution function F_X. We normalize the WTP X_t and the prices $r_{t,c}$ to $[0,1]$ and a customer buys an item if and only if $X_t \geq r_{t,c}$, i.e. the probability of a sale at price $r_{t,c}$ is $p(r_{t,c}) = 1 - F_X(r_{t,c})$. In line with standard assumptions from literature (see, e.g., Ziya et al. 2004 or Talluri and van Ryzin 2004), we assume that $p(r_{t,c})$ is a continuous, strictly decreasing, and twice continuously differentiable function, and the revenue function $r_{t,c} \cdot p(r_{t,c})$ is strictly concave. Thus, the following properties hold:

P.1. $p(r_{t,c})$ is twice continuously differentiable

P.2. $p'(r_{t,c}) < 0$

P.3. $0 > \frac{d^2}{d\, r_{t,1}^2}\left(r_{t,c} \cdot p(r_{t,c})\right) = 2p'(r_{t,c}) + r_{t,c} \cdot p''(r_{t,c})$

P.4. $p(0) = 1$ and $p(1) = 0$.

These conditions are met, for example, by the uniform distribution, and ensure that the revenue function is unimodal and the pricing problem is well behaved.

3.2 Risk-neutral dynamic pricing

In the classical risk-neutral case, the firm maximizes the total expected revenue $V_{T,C}^N$ over the selling horizon:

$$V_{T,C}^N = \max_{\substack{r_{t,c} \\ \forall\, t\in\{1,\dots,T\}, c\in\{1,\dots,C\}}} \mathbb{E}\left[\sum_{t=1}^{T} r_{t,c} \cdot 1_{\{X_t \geq r_{t,c}\}}\right] \tag{1}$$

Please note that only $r_{t,c}\ \forall\, t = 1, \dots, T\ \wedge c = 1, \dots, C$ are decision variables and we set $r_{t,0} = 1$ to avoid selling an additional unit when the stock is depleted. As the expected value is a linear function, a stage-wise optimization with the Bellman equation can be used to calculate $V_{T,C}^N$:

$$V_{t,c}^N = \max_{r_{t,c}} \left\{ p(r_{t,c}) \cdot \left(r_{t,c} + V_{t-1,c-1}^N\right) + \left(1 - p(r_{t,c})\right) \cdot V_{t-1,c}^N \right\} \tag{2}$$

Here, $V_{t,c}^N$ denotes the optimal expected revenue-to-go from period t onwards until period 0, with a remaining stock of c units. In period t, this is an expectation over two events. With probability $p(r_{t,c})$, a sale occurs and the firm collects a revenue of $r_{t,c}$. In addition, the firm expects a revenue of $V_{t-1,c-1}^N$ with a reduced stock of $c-1$ units from the next period onwards. With probability $1 - p(r_{t,c})$, no sale occurs and the firm expects a revenue of $V^N_{t-1,c}$ from stock c. Two boundary conditions ensure termination of the recursion and the sale of less than or equal to C items: $V^N_{0,c} = 0$ for $c \geq 0$ and $V^N_{t,c} = -\infty$ for $c < 0$.

Note that the primary goal of solving (2) is usually not only to calculate the expected revenue, but to obtain a policy. Such a decision rule indicates, for every state (c, t), the price $r_{t,c}$ to post.

3.3 General representations of CVaR

3.3.1. CVaR in a static setting

Given a probability level $\alpha \in [0,1]$ and a random variable R denoting a profit with a distribution function $F_R(y) = \mathbb{P}(R \leq y)$, the VaR is simply the α-quantile ($VaR_\alpha(R) = F_R^{-1}(\alpha)$). Intuitively, CVaR can be thought of as the expectation below VaR_α or the α-quantile:

$$\mathrm{CVaR}_\alpha(R) = \mathbb{E}[R : R \leq F_R^{-1}(\alpha)]. \tag{3}$$

Note that (3) is equivalent to the formal definition only when considering probability spaces without atoms. However, revenue is clearly a discrete random variable for each given policy in dynamic pricing. Thus, we turn to CVaR's dual representation for a definition (e.g., Pflug and Pichler 2016):

$$\mathrm{CVaR}_\alpha(R) = \inf_Z \{\mathbb{E}[RZ] : \mathbb{E}[Z] = 1,\ 0 \leq Z \leq 1/\alpha\} \tag{4}$$

with $\mathrm{CVaR}_0(R) = \operatorname{ess\,inf} R$. In (4), the infimum is over all nonnegative random variables $Z \geq 0$ with expectation $\mathbb{E}[Z] = 1$ (densities), which satisfy the additional truncation constraint $Z \leq 1/\alpha$. Note that (3) is also known as the Tail Conditional Expectation and (4) as the Expected Shortfall. Both can be applied to discrete distributions and are equivalent for continuous distributions.

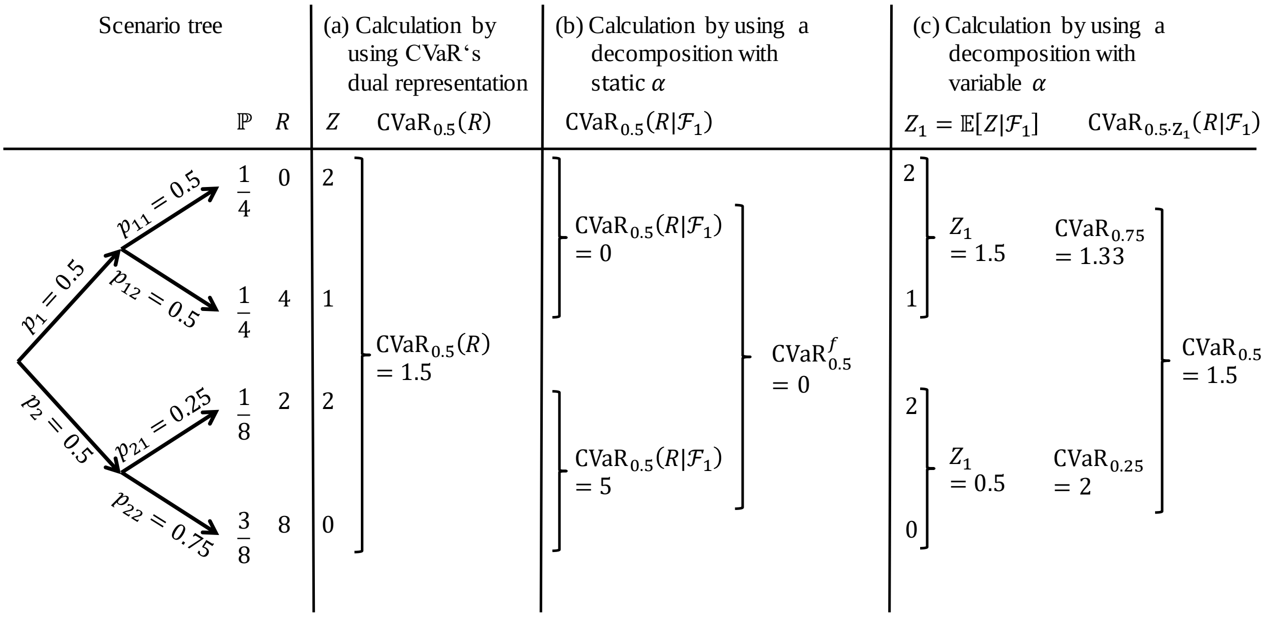

Figure 1: Comparison of three variants to calculate CVaR in a multistage setting

Figure 1 illustrates CVaR's calculation with (4) for an example. We first ignore the scenario tree on the left and directly consider *CVaR of the outcomes at the final stage* (Figure 1 (a)): The random variable R with four realizations and associated probability measure $\mathbb{P}$. Now, note that for discrete events, (4) is essentially a continuous knapsack problem and the lowest (worst) outcomes of R are subsequently assigned Z-values as high as possible until $\mathbb{E}[Z] = 1$ holds. Then, $\mathrm{CVaR}_{0.5}(R) = \mathbb{E}[RZ] = 1.5$ follows. Thus, the Z can intuitively be viewed as weights that indicate whether an event (atom) falls below $F_R^{-1}(\alpha)$, and is thus included in the CVaR's expectation. Roughly, (4) differs from (3) in the sense that the atom with R-value 4, whose value of the cumulative density function stretches over α, is divided and only partially included in CVaR's expectation.

3.3.2. CVaR in a multistage setting

Next, we consider how CVaR can be calculated in a multistage setting and directly illustrate this using the example in Figure 1. Therefore, we consider the scenario tree on the left of Figure 1 and explore how the fact that R realizes in a two-stage process can be used for a recursive (or nested) calculation.

Analogous to the nested calculation of the expected value, a straightforward approach seems the *nested calculation of CVaR at a fixed probability level*: $\mathrm{CVaR}_{\alpha}^{f}(R|\mathcal{F}_t) = \mathrm{CVaR}_{\alpha}(\mathrm{CVaR}_{\alpha}(R|\mathcal{F}_{t-1})|\mathcal{F}_t)$ where $\mathcal{F}_t$ denotes a sigma-algebra in period t of the stochastic process visualized by the tree structure ($\mathcal{F}_t \subset \mathcal{F}_{t-1}$). In the example (Figure 1 (b)), we first calculate $\mathrm{CVaR}_{0.5}$ for the upper and lower subtree, and then use the results to calculate $\mathrm{CVaR}_{0.5}$ at the root. We obtain $\mathrm{CVaR}_{0.5}(R) = 0$ and $\mathrm{CVaR}_{0.5}(R) = 5$ for the subtrees, respectively. At the root, each subtree has a probability of 0.5, and, accordingly, only the upper subtree (with lower CVaR) is considered. We obtain $CVaR_{0.5}^{f}(R) = \mathrm{CVaR}_{0.5}(\mathrm{CVaR}_{0.5}(R|\mathcal{F}_1)|\mathcal{F}_2) = 0$. Comparing this with the previous subsection (Figure 1 (a)), 0 is obviously not the correct $\mathrm{CVaR}_{0.5}$ of R. This illustrates that a nested calculation of CVaR with a fixed α is difficult to interpret and certainly not what decision makers would understand under multistage risk (Pflug and Pichler 2016, Rudloff et al. 2014). The reason is that in each subtree, the share of outcomes that are below $F_R^{-1}(\alpha)$ and, thus, have to be included in CVaR's expectation is inherently unknown and usually not equal to the initial $\alpha = 0.5$.

It is indeed possible to recursively calculate the CVaR of a multistage process' final outcome (Pflug and Pichler 2016). Key is a property that obviously holds at the root: The probability level reflects the share of outcomes that are below $F_R^{-1}(\alpha)$ and therefore included in the CVaR. The preservation of this property requires relaxing the assumption of a fixed probability level α at intermediate periods. Instead, as new information becomes available, the probability level in period t is now adapted by the random variable Z_t such that the modified probability level at each node still reflects the share of outcomes that are below $F_R^{-1}(\alpha)$. The *nested calculation of CVaR at random probability level* is given by

$$\mathrm{CVaR}_{\alpha}(R|\mathcal{F}_t) = \inf_{Z_t}\left\{\mathbb{E}\left[Z_t \cdot \mathrm{CVaR}_{\alpha\cdot Z_t}(R|\mathcal{F}_{t-1})\right]: \mathbb{E}[Z_t] = 1,\ 0 \le Z_t \le 1/\alpha\right\} \quad (5)$$

with $\mathrm{CVaR}_0(R) = \operatorname{ess\,inf} R$. Moreover, if Z is the optimal dual density for (4), then $Z_t = \mathbb{E}[Z|\mathcal{F}_t]$.

Figure 1 (c) illustrates this for our two-stage example. According to (5), only respectively 75% and 25% of the upper and lower subtree's probability mass are included in CVaR's expectation. This exactly represents the shares included in the direct calculation of the final outcomes' CVaR in the previous subsection. We obtain $\mathrm{CVaR}_{0.75}(R) = 1.33$ and $\mathrm{CVaR}_{0.25}(R) = 2$ for the upper and lower subtree, respectively, and $\mathrm{CVaR}_{0.5}(R) = \mathrm{CVaR}_{0.5}\left(\mathrm{CVaR}_{\alpha\cdot Z_1}(R|\mathcal{F}_1)|\mathcal{F}_2\right) = 1.5$ at the root. Different from our illustration, the Z-values of the tree's leafs and, thus, the probability levels $\mathbb{E}[Z|\mathcal{F}_1] \cdot \alpha$ in period 1 are not known a priori but come along with the solution of the entire problem.

3.4 Problem statement

The firm maximizes CVaR at a given probability level α of the total revenue obtained over the selling horizon. This can be formally stated as follows:

$$\tilde{V}_{T,C}(\alpha) = \max_{\substack{r_{t,c} \\ \forall\, t\in\{1,\dots,T\},c\in\{1,\dots,C\}}} \mathrm{CVaR}_\alpha \left(\sum_{t=1}^{T} r_{t,c} \cdot 1_{\{X_t \geq r_{t,c}\}}\right) \tag{6}$$

As in Section 3.2, we require $r_{t,0} = 1$ to avoid selling an additional unit when the stock is depleted.

3.5 Recursive maximization of CVaR in dynamic pricing

Building on (5), we now model the risk-averse dynamic pricing problem using a recursive formulation. Pflug and Pichler (2016) show that (5) can be substituted for the expectation in a general Bellman Equation under conditions that hold for the dynamic pricing problem as defined in Subsections 3.1 and 3.2: $\sum_{t=1}^{T} r_{t,c} \cdot 1_{\{X_t \geq r_{t,c}\}}$ is random upper semi-continuous in $r_{t,c}$ and X and X evaluates in some convex, compact subset of $\mathbb{R}^n$. Thus, at each stage t, the following problem has to be solved:

$$\tilde{V}_{t,c}(\alpha) = \max_{r_{t,c}} \mathrm{CVaR}_\alpha \left(1_{\{X_t \geq r_{t,c}\}} \cdot \left(r_{t,c} + \tilde{V}_{t-1,c-1}\left(z_{t-1,c-1} \cdot \alpha\right)\right) + 1_{\{X_t < r_{t,c}\}} \cdot \tilde{V}_{t-1,c}\left(z_{t-1,c} \cdot \alpha\right)\right)$$

where $z_{t-1,c}$ and $z_{t-1,c-1}$ are the values of the optimal solution to the minimization calculating CVaR (see (5)). We obtain the following Bellman equation to calculate $\tilde{V}_{T,C}(\alpha)$ for $\alpha \in (0,1]$:

$$\begin{aligned}\tilde{V}_{t,c}(\alpha) = \max_{r_{t,c}} \min_{z_{t-1,c-1}, z_{t-1,c}} \Big\{ & p(r_{t,c}) \cdot z_{t-1,c-1} \cdot \left(r_{t,c} + \tilde{V}_{t-1,c-1}\left(\alpha \cdot z_{t-1,c-1}\right)\right) \\ & + \left(1 - p(r_{t,c})\right) \cdot z_{t-1,c} \cdot \tilde{V}_{t-1,c}\left(\alpha \cdot z_{t-1,c}\right)\Big\}\end{aligned} \tag{7}$$

subject to

$$1 = \left(1 - p(r_{t,c})\right) \cdot z_{t-1,c} + p(r_{t,c}) \cdot z_{t-1,c-1}$$

$$z_{t-1,c},\ \ z_{t-1,c-1} \leq \frac{1}{\alpha}$$

$$r_{t,c} \in [0,1]$$

with the boundary conditions $\tilde{V}_{t,c}(\alpha) = -\infty$ for $c < 0$ and $\tilde{V}_{0,c}(\alpha) = 0$ for $c \geq 0$. For $\alpha = 0$, we have $\tilde{V}_{t,c}(0) = 0$. As the probability level is adapted in every time period it is included in the state space yielding a dynamic program with uncountable infinite state and action space.

Whereas $\tilde{V}_{t,c}(\alpha)$ defined in (7) exhibits no special structure apart from being monotonous, we reformulate it to obtain an equivalent value function $V_{t,c}(\alpha)$ that is monotonous and convex (see Ogryczak and Ruszczynski 2002), and is thus easier to handle. We respectively substitute $\alpha_{t-1,c}$ and $\alpha_{t-1,c-1}$ for $\alpha \cdot z_{t-1,c}$ and $\alpha \cdot z_{t-1,c-1}$, and change the objective from CVaR_α to $\alpha \cdot \mathrm{CVaR}_\alpha$. These modifications lead to the following Bellman equation for $\alpha \in (0,1]$:

$$\begin{aligned}V_{t,c}\left(\alpha_{t,c}\right) = \max_{r_{t,c}} \min_{\alpha_{t-1,c}, \alpha_{t-1,c-1}} \Big\{ & p(r_{t,c}) \cdot \left(V_{t-1,c-1}\left(\alpha_{t-1,c-1}\right) + r_{t,c} \cdot \alpha_{t-1,c-1}\right) \\ & + \left(1 - p(r_{t,c})\right) \cdot V_{t-1,c}\left(\alpha_{t-1,c}\right)\Big\}\end{aligned} \tag{8}$$

subject to

$$\alpha_{t,c} = \left(1 - p(r_{t,c})\right) \cdot \alpha_{t-1,c} + p(r_{t,c}) \cdot \alpha_{t-1,c-1} \tag{9}$$

$$\alpha_{t-1,c},\ \ \alpha_{t-1,c-1},\ \ r_{t,c} \in [0,1] \tag{10}$$

with $V_{t,c}(\alpha) = -\infty$ for $c < 0$ and $V_{0,c}(\alpha) = 0$ for $c \geq 0$. $V_{t,c}(\alpha) = \alpha \cdot \tilde{V}_{t,c}(\alpha)$ denotes the value of $\alpha \cdot \mathrm{CVaR}_\alpha$ when following an optimal policy from period t onwards.

Remark 1 Considering the special case of $\alpha_{t,c} = 0$, a feasible solution of the inner minimization problem is $\alpha_{t-1,c-1} = \alpha_{t-1,c} = 0$ and, therefore, $V_{t,c}(0) = 0\ \forall\, r_{t,c}$. Without loss of generality, we then choose $r_{t,c} = 0\ \forall\, t, c$. As this special case is trivial, we exclude the case for the rest of the work and consider only $\alpha_{t,c} > 0$.

Backward induction, the classical solution approach for dynamic programs, can only handle dynamic programs with a discrete state space, but the dynamic program (8) possesses a partially continuous state space. Moreover, there is a continuous action space and no analytical solution available for the optimization problem solved on each stage. In this case, the standard approach is to discretize both by a finite set of points. As this always involves a trade-off between a rough discretization and long runtimes, we develop problem-specific approaches. In the following Section 4, we analytically investigate and solve (8) for $C = 1$. In Section 5, we present two solution algorithms for arbitrary capacities, each requiring only to discretize the state or the action space.

4 Structural properties for $C = 1$

If there is only one unit of capacity for sale, the value function given in formula (8) simplifies to

$$V_{t,1}(\alpha_{t,1}) = \max_{r_{t,1}} \min_{\alpha_{t-1,1},\alpha_{t-1,0}} \left\{ \left(1 - p(r_{t,1})\right) \cdot V_{t-1,1}(\alpha_{t-1,1}) + p(r_{t,1}) \cdot r_{t,1} \cdot \alpha_{t-1,0} \right\} \tag{11}$$

subject to

$$\alpha_{t,1} = \left(1 - p(r_{t,1})\right) \cdot \alpha_{t-1,1} + p(r_{t,1}) \cdot \alpha_{t-1,0} \tag{12}$$

$$\alpha_{t-1,1},\ \ \alpha_{t-1,0},\ \ r_{t,1} \in [0,1] \tag{13}$$

Our presentation proceeds as follows: In Section 4.1, we give sufficient conditions for solutions of the optimization problems inherent in (11) and derive an explicit recursive formulation of the value function dependent on $V_{t-1,1}(\alpha_{t-1,1})$. In Section 4.2, we state certain properties of the optimal price, and in Section 4.3 we further refine the recursive formulation of the value function to a formula no longer dependent on $V_{t-1,1}(\alpha_{t-1,1})$.

4.1 Analytical formulation of the value function

The following proposition characterizes the value function $V_{t,1}(\alpha_{t,1})$, the optimal price $r_{t,1}(\alpha_{t,1})$, and the optimal probability level $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$ as, respectively, functions of $\alpha_{t,1}$ and $r_{t,1}(\alpha_{t,1})$. Given $V_{t-1,1}(\alpha_{t-1,1})$, these quantities are determined by solving the optimization problem (11) analytically.

Lemma 1 *For all $\alpha_{t,1} > 0$ with $t \in \{0, \dots, T\}$, $r_{t,1}(\alpha_{t,1}) \notin \{0,1\}$. Thus, $r_{t,1} \in [0,1]$ can be replaced by $r_{t,1} \in (0,1)$.*

Proposition 1 *Let $\alpha_{t,1}^{PI_{t-1}} = p'\left(V'_{t-1,1}(1)\right) \cdot \left(V_{t-1,1}(1) - V'_{t-1,1}(1)\right) + 1 - p\left(V'_{t-1,1}(1)\right)$. The domain of $\alpha_{t,1}$, $[0,1]$, consists of two intervals $\left[0, \alpha_{t,1}^{PI_{t-1}}\right]$ and $\left(\alpha_{t,1}^{PI_{t-1}}, 1\right]$ for which $V_{t,1}(\alpha_{t,1})$, $r_{t,1}(\alpha_{t,1})$ and $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$ are characterized as follows.*

Given $r_{t,1}(\alpha_{t,1})$, the optimal level $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$ is:

$$\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right) = \begin{cases} \left(V'_{t-1,1}\right)^{-1}\left(r_{t,1}(\alpha_{t,1})\right), & \text{if } \alpha_{t,1} \in [0, \alpha_{t,1}^{PI_{t-1}}] \\ 1, & \text{if } \alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1] \end{cases} \tag{14}$$

and $\alpha_{t-1,0}$ has to be chosen so that the condition $\left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right) + p\left(r_{t,1}(\alpha_{t,1})\right) \cdot \alpha_{t-1,0} = \alpha_{t,1}$ is satisfied.

The optimal price $r_{t,1}(\alpha_{t,1})$ is implicitly given by the sufficient conditions:

$$\begin{cases} \left(\alpha_{t,1} - p'(r_{t,1}) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}(r_{t,1})\right) - r_{t,1} \cdot \alpha_{t-1,1}(r_{t,1})\right) \right. \\ \left. - \left(1 - p(r_{t,1})\right) \cdot \alpha_{t-1,1}(r_{t,1}) = 0\right) \wedge \left(r_{t,1} \in \left(0, V'_{t-1,1}(1)\right]\right) & \text{if } \alpha_{t,1} \in [0, \alpha_{t,1}^{PI_{t-1}}] \\ \alpha_{t,1} - p'(r_{t,1}) \cdot \left(V_{t-1,1}(1) - r_{t,1}\right) - \left(1 - p(r_{t,1})\right) = 0 & \text{if } \alpha_{t,1} \in \left(\alpha_{t,1}^{PI_{t-1}}, 1\right] \end{cases} \tag{15}$$

The value function is given by:

$$V_{t,1}(\alpha_{t,1}) = \begin{cases} \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) \right. \\ \left. - r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) + r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t,1} & \text{if } \alpha_{t,1} \in [0, \alpha_{t,1}^{PI_{t-1}}] \\ \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \left(V_{t-1,1}(1) - r_{t,1}(\alpha_{t,1})\right) + r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t,1} & \text{if } \alpha_{t,1} \in \left(\alpha_{t,1}^{PI_{t-1}}, 1\right] \end{cases} \tag{16}$$

Proof. The proof is based on an analytical solution of (11) and given in Online Supplement S.2. There, we also show that $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$, $\alpha_{t-1,0}$ and $r_{t,1}(\alpha_{t,1})$ are well defined.

4.2 Properties of the optimal price

The following propositions provide properties of the optimal price. The proofs are given in Online Supplement S.3 – S.8.

Proposition 2 *The optimal price $r_{t,1}(\alpha_{t,1})$ is strictly monotonically increasing in $\alpha_{t,1} \in [0,1]$.*

Proposition 3 *The optimal price $r_{t,1}(\alpha_{t,1})$ is continuous in the probability level $\alpha_{t,1} \in [0,1]$. Moreover, $\lim_{\alpha_{t,1} \searrow 0} r_{t,1}(\alpha_{t,1}) = 0$.*

Together, these propositions imply that for every feasible price $r_{t,1}$ lower than the price of the expected value optimal policy, i.e. $r_{t,1}(1)$, there is an $\alpha_{t,1}$ so that $r_{t,1}$ is optimal for $\alpha_{t,1}$. The strict monotonicity of $r_{t,1}(\alpha_{t,1})$ is also as intuitively expected, because a lower risk aversion, i.e. a higher $\alpha_{t,1}$, leads to a higher price and, therefore, the risk of no sale increases, as does the possible revenue.

Proposition 4 *Let $\alpha_{t,1}$ be the probability level in period t and $\alpha_{t-1,1}$ be the probability level in $t-1$ which occurs when there are no sales. Then it holds that:*

$$\alpha_{t,1} \leq \alpha_{t,1}^{PI_{t-1}} \Rightarrow r_{t-1,1}(\alpha_{t-1,1}) = r_{t,1}(\alpha_{t,1}),$$
$$\alpha_{t,1} \geq \alpha_{t,1}^{PI_{t-1}} \Rightarrow r_{t-1,1}(\alpha_{t-1,1}) = r_{t-1,1}(1) = r_{t,1}(\alpha_{t,1}^{PI_{t-1}}) \leq r_{t,1}(\alpha_{t,1}),$$

and, thus, the optimal price $r_{t,1}(\alpha_{t,1})$ is nonincreasing over time (i.e. nondecreasing in t).

Together with Propositions 1-3, Proposition 4 shows that the structure of the optimal risk-averse policy is fundamentally different from the structure usually found in the standard setting of risk-neutral dynamic pricing. When selling one unit of capacity and optimizing the Conditional Value-at-Risk, a fixed price policy is optimal at the beginning of the selling horizon until $\alpha_{t,1} \geq \alpha_{t,1}^{PI_{t-1}}$. Hereafter, the price follows the expected value optimal policy and $\alpha_{t',1} = 1$ for $t' < t$. The lower the initial probability level, the later the switch to the expected value-optimal (dynamic) policy.

Propositions 2 and 4 are illustrated in Figure 2, using an example with $T = 10$ time periods and uniform WTP ($p(r_{t,c}) = 1 - r_{t,c}$). It shows the optimal prices set in each time period if no sale occurs, with different initial probability levels $\alpha_{10,1}$. The risk-neutral optimal price is initially $r_{10,1} = 0.86$ in $t = 10$ and declines over time. In contrast, the risk-averse optimal price for $\alpha_{10,1} = 0.8$ is initially $r_{10,1} = 0.81$ and remains constant until period $t = 7$. From period $t = 6$ onwards, it equals the risk-neutral price in the respective period and, thus, declines. The optimal price for $\alpha_{10,1} = 0.4$ is initially even lower, with $r_{10,1} = 0.74$, and remains constant until period $t = 4$.

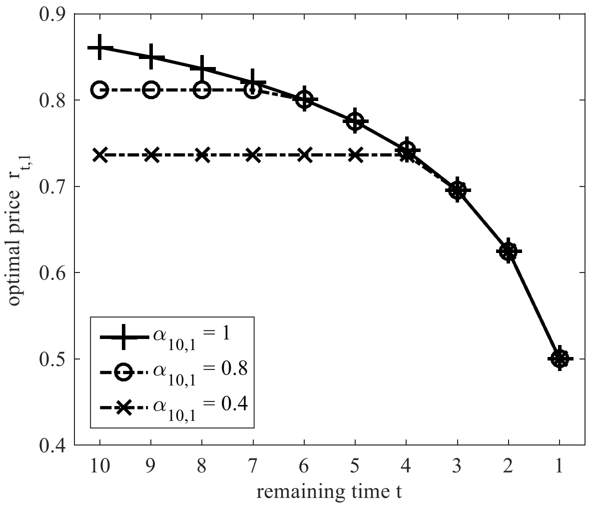

Figure 2: Price process over time

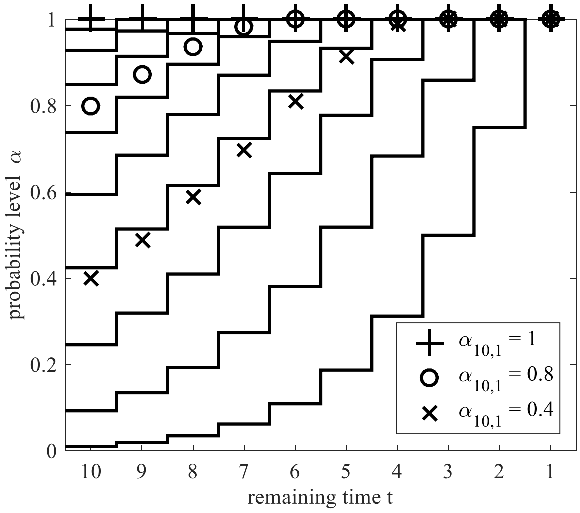

Figure 3: Intervals and probability levels

The following propositions characterize the optimal policy and the evolution of the probability level.

Proposition 5 *The Value-at-Risk at level $\alpha_{t,1}$ in period t is equal to the optimal price $r_{t,1}(\alpha_{t,1})$.*

Proposition 5 is a direct consequence of CVaR_α's definition as the expectation below the VaR. It is quite intuitive. As there is at most a single sale, increasing the price above the VaR does not contribute to CVaR of total revenue, but just decreases the probability of a sale and, thus, wastes time. On the other hand, we already know that the price is nonincreasing. Thus, setting a price strictly below the VaR is a contradiction as it implies that total revenue is at most this lower price and, thus, VaR would be at most this lower price.

Proposition **6** *The probability level $\alpha_{t,1} \in [0,1]$ is monotonically increasing over time, i.e. $\alpha_{t,1} \leq \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$.*

Remember the discussion of the stochastic nature of intermediate probability levels from Section 3.3. With every time period passing, the set of possible future events shrinks. If no sale occurs, things are going bad and the decision maker adjusts his preferences. More precisely, the chances of obtaining a revenue in the upper $(1 - \alpha_{t,1})$-part of the distribution diminish and, thus, a higher share of the still (in $t - 1$) possible events is included in CVaR's expectation, which is still conditional on revenue being below the (former, unchanged) VaR_{α_t}.

4.3 Transformation from sufficient conditions to a recursive formula

We now state a formula for the value function which is not recursive in the sense that $V_{t,1}$ depends on $V_{t-1,1}$, but is given explicitly for intervals of $\alpha_{t,1} \in [0,1]$, evolving over time. For notational convenience, we only consider the special case of uniformly distributed WTPs, i.e. $p(r_{t,1}) = 1 - r_{t,1}$, in the following.

Proposition 7 *Let r^{PI_t} be defined as*

$$r^{PI_t} = \frac{1}{2} + \frac{\left(r^{PI_{t-1}}\right)^2}{2} \text{ and } r^{PI_0} = 0. \tag{17}$$

Furthermore, we partition $(0,1]$ into intervals $S_{t,1}^{j} = \left(\alpha_{t,1}^{PI_{j-1}}, \alpha_{t,1}^{PI_j}\right]$ indexed by $0 < j \leq t$ with:

$$\alpha_{t,1}^{PI_j} = \begin{cases} 2r^{PI_j} \cdot \alpha_{t-1,1}^{PI_j} - \left(r^{PI_j}\right)^2 \cdot \alpha_{t-2,1}^{PI_j} & \text{if } 0 < j < t \\ 1 & \text{if } j \geq t \end{cases} \quad \textit{and } \alpha_{t,1}^{PI_0} = 0 \tag{18}$$

Moreover, $\alpha_{t,1}^{PI_j} = 1$ for $t \leq 0$. Then,

- *the optimal risk-neutral price $r_{t,1}(1)$ in period t is given by $r_{t,1}(1) = r^{PI_t}$ and* (19)
- *for $\alpha_{t,1} \in S_{t,1}^{j}$ with $j \leq t$ holds $V_{t,1}(\alpha_{t,1}) = -r_{t,1}^{t-j+2} + r_{t,1}^{t-j+1} \cdot V_{j-1,1}(1) + r_{t,1} \cdot \alpha_{t,1}$* (20)
- *where the price $r_{t,1}$ is implicitly defined by $-(t-j+2) \cdot r_{t,1}^{t-j+1} + (t-j+1) \cdot r_{t,1}^{t-j} \cdot V_{j-1,1}(1) + \alpha_{t,1} = 0$* (21)
- *and $r_{t,1}\left(\alpha_{t,1}^{PI_j}\right) = r^{PI_j}, \forall j \leq t$.* (22)
- *For $t \geq 2$, we have $\alpha_{t,1} \in S_{t,1}^{j} \Rightarrow \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right) \in S_{t-1,1}^{min\{j,t-1\}}$.* (23)

Proof. The proof is given in Online Supplement S.10.

This proposition provides a possibility to directly calculate the value function for a given time period t. In addition, it also shows that if $\alpha \in S_{t,1}^{j}$, then $\alpha_{t,1}$ reaches 1 in period $j-1$, and, thus, the risk-neutral price is optimal from period $j-1$ onwards. Together with Proposition 3, this shows that the price remains constant until period $t = j$.

Figure 3 illustrates this by again using the previous example ($T = 10$, $p(r_{t,1}) = 1 - r_{t,1}$), and shows the intervals defined by (18). Obviously, there is only one interval in $t = 1$, and for each additional time period the existing intervals become smaller, and a new interval between 1 and the existing intervals is added. In addition, the figure illustrates the meaning of the intervals beyond the formal definition and usage to calculate the value function and optimal price according to (20) and (21). Therefore, it shows the three sales processes from Figure 2 with the evolution of the corresponding level $\alpha_{t,1}$ over time. The level remains 1 for the risk-neutral sales process ($\alpha_{10,1} = 1$). The level is initially $\alpha_{10,1} = 0.8$ for the first risk-averse sales process and, thus, in the 7th interval. Subsequently, the level $\alpha_{t,1}$ increases, but always remains in the 7th interval. Note that until period $t = 7$, the price remains constant (see Figure 2), although the level $\alpha_{t,1}$ increases. When the interval ceases to exist in period $t = 6$, the corresponding level $\alpha_{6,1}$ reaches 1 and remains there. Hereafter, the risk-neutral price is optimal. The explanation regarding the third sales process with $\alpha_{10,1} = 0.4$ is completely analogous, except that the level is in the 4th interval and reaches 1 in period $t = 3$.

5 Solution algorithms for arbitrary capacity

In this section, we present two different algorithms for the solution of the dynamic program developed in Section 3. Both algorithms solve the dynamic program by backward induction, that is, analogously to the recursive formulation of (8), the value function in period $t-1$ is used to calculate the value function in period t. The first algorithm uses a discrete state space obtained by discretizing the probability level α and a continuous action space. The second algorithm uses the original, partially continuous state space and a discretized action space, that is, discrete prices. We focus on uniformly distributed WTP, i.e. $p(r_{t,c}) = 1 - r_{t,c}$, although the algorithms can be generalized to other WTPs.

5.1 Algorithm A (discrete state space)

In this algorithm, we discretize only the state space while the action space remains continuous and determine the optimal price for each discrete probability level α considered. The basic idea is to successively consider the grid points, and to use the optimal price determined at the previous grid point as the starting point for the search for the optimal price in the next grid point. At each grid point, we replace the difficult bilevel problem (8) involving piecewise linear functions with a sequence of simpler optimization problems. The resulting subproblems only involve affine linear functions.

5.1.1. Overview

We discretize the domain $[0,1]$ of the probability level α by a finite set of points $0 = \alpha^1 \leq \alpha^2 \leq \cdots \leq \alpha^{n_\alpha} = 1$, denoted by $\mathcal{A} = \{\alpha^1, \dots, \alpha^{n_\alpha}\}$. As a result of this discretization of α, the value function $V_{t,c}^A(\alpha)$ is piecewise linear and also convex in α (Pflug and Pichler 2016). Therefore, the inner minimization – which determines $\alpha_{t-1,c}$ and $\alpha_{t-1,c-1}$ for given $\alpha^i \in \mathcal{A}$ and $r_{t,c}$ – becomes a continuous knapsack problem, which can be efficiently solved with a simple greedy procedure (Gönsch and Hassler 2014).

In particular, there are $2(n^a - 1)$ items available. The set $\mathcal{J}_{t-1,c} = \{1, 2, \dots, n^a - 1\}$ contains the items related to $V_{t-1,c}^A(\cdot)$ and each item $j \in \mathcal{J}_{t-1,c}$ has utility $u_{t-1,c}^j = \left(1 - p(r_{t,c})\right)\left(V_{t-1,c}^A(\alpha^{j+1}) - V_{t-1,c}^A(\alpha^j)\right)$ and weight $w_{t-1,c}^j = \left(1 - p(r_{t,c})\right)\left(\alpha^{j+1} - \alpha^j\right)$. Decision variable $x_{t-1,c}^j \in [0,1]$ denotes the amount selected from item j. Analogously, the set $\mathcal{J}_{t-1,c-1} = \{1, 2, \dots, n^a - 1\}$ contains the items related to $V_{t-1,c-1}^A(\cdot)$ and each item $j \in \mathcal{J}_{t-1,c-1}$ has utility $u_{t-1,c-1}^j = p(r_{t,c}) \cdot \left(V_{t-1,c-1}^A(\alpha^{j+1}) - V_{t-1,c-1}^A(\alpha^j) + r_{t,c} \cdot (\alpha^{j+1} - \alpha^j)\right)$, weight $w_{t-1,c-1}^j = p(r_{t,c}) \cdot (\alpha^{j+1} - \alpha^j)$ and decision variable $x_{t-1,c-1}^j \in [0,1]$. Slightly different from the common knapsack problem, the goal is to pack a knapsack with minimal utility and a weight equal to α^i. Thus, the problem formulation is:

$$V_{t,c}^A(\alpha^i) = \min_{x_{t-1,c}^j, x_{t-1,c-1}^j \forall j} \left\{\sum_{j=1}^{n^a-1} x_{t-1,c}^j u_{t-1,c}^j + x_{t-1,c-1}^j u_{t-1,c-1}^j\right\} \tag{24}$$

subject to

$$\sum_{j=1}^{n^a-1} x_{t-1,c}^j w_{t-1,c}^j + x_{t-1,c-1}^j w_{t-1,c-1}^j \geq \alpha^i$$

$$x_{t-1,c}^j, x_{t-1,c-1}^j \in [0,1] \qquad \forall j \in \mathcal{J}_{t-1,c}, \mathcal{J}_{t-1,c-1}$$

To solve this knapsack problem, we sort the relative utilities $u_{t-1,\cdot}^j / w_{t-1,\cdot}^j$ in increasing order and add the corresponding items to the knapsack with a simple greedy procedure. As the relative utilities are related to the slopes of the piecewise linear and convex functions $V_{t-1,\cdot}^A(\cdot)$ between α^j and α^{j+1}, they are already sorted and increasing in j for both $\mathcal{J}_{t-1,c}$ and $\mathcal{J}_{t-1,c-1}$. As there is always one optimal solution of a continuous knapsack with at most one product split (neither fully included or excluded), at least either $\alpha_{t-1,c}^*$ or $\alpha_{t-1,c-1}^*$ is located in a grid point (i.e. $\alpha_{t-1,c}^* \in \mathcal{A}$ or $\alpha_{t-1,c-1}^* \in \mathcal{A}$). (Formally, equation (24) contains terms of the form of $V_{t-1,\cdot}^A(\alpha^i) - V_{t-1,\cdot}^A(0)$, but $V_{t-1,\cdot}^A(0) = 0$ (see Section 3.5).)

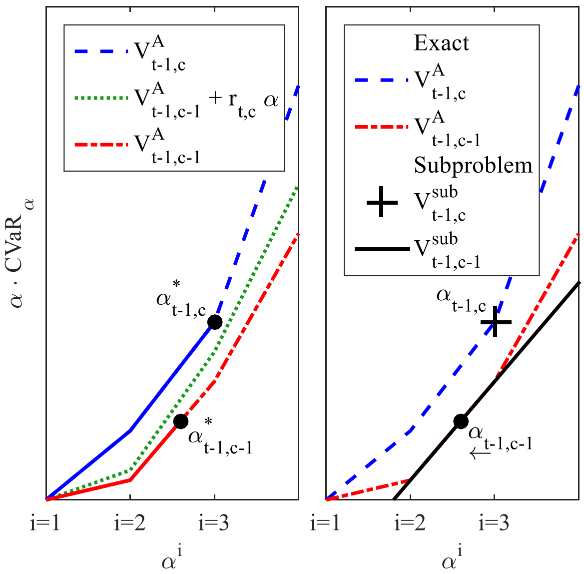

Figure 4: Illustration of the greedy procedure (left) and of the subproblem (right)

Figure 4 (left part) shows an example to illustrate this. Let $s^j_{t-1,c}$ and $s^j_{t-1,c-1}$ denote the slope of the piecewise linear functions $V^A_{t-1,c}(\cdot)$ and $V^A_{t-1,c-1}(\cdot)$ between α^j and α^{j+1}, i.e. $s^j_{t-1,c} = \left(V^A_{t-1,c}(\alpha^{j+1}) - V^A_{t-1,c}(\alpha^j)\right)/(\alpha^{j+1} - \alpha^j) = u^j_{t-1,c}/w^j_{t-1,c}$ and $s^j_{t-1,c-1} + r_{t,c} = u^j_{t-1,c-1}/w^j_{t-1,c-1}$ for $j = 1, \dots, n_\alpha - 1$, and set $s^{n_\alpha}_{t-1,c} = s^{n_\alpha}_{t-1,c-1} = \infty$. In the figure, we have $s^1_{t-1,c-1} + r_{t,c} < s^1_{t-1,c} < s^2_{t-1,c} < s^2_{t-1,c-1} + r_{t,c} < s^3_{t-1,c}$, and the $\alpha_{t-1,\cdot}$ are increased by the greedy procedure in this sequence until constraint (9) holds. The procedure starts with $\alpha_{t-1,c} = \alpha_{t-1,c-1} = 0$ and first sets $\alpha_{t-1,c-1} = \alpha^2$, then $\alpha_{t-1,c} = \alpha^2$, and then $\alpha_{t-1,c} = \alpha^3$. Now $\alpha_{t-1,c-1}$ must be increased again. However, it is only increased to $\alpha^*_{t-1,c-1}$, because condition (9) holds in this example, and we have found the optimal values $\alpha^*_{t-1,c}$ and $\alpha^*_{t-1,c-1}$. Thus, we have $\alpha^*_{t-1,c} \in \mathcal{A}$ and $\alpha^*_{t-1,c-1} \notin \mathcal{A}$.

We now formulate Algorithm A using this characteristic of the minimization's solution. The algorithm is consecutively executed to consider the grid points $\alpha^i \in \mathcal{A}, i = 1, \dots, n_\alpha$ with an increasing probability level. At each grid point α^i, a search for the optimal price $r^*_{t,c}(\alpha^i)$ is performed, starting at the optimal price determined at the previous grid point ($r_{t,c} = r^*_{t,c}(\alpha^{i-1})$). The price $r_{t,c}$ is now iteratively increased until the optimal price $r^*_{t,c}(\alpha^i)$ is attained. More precisely, we increase the price $r_{t,c}$ and simultaneously modify the vector $\left(\alpha^*_{t-1,c}(\alpha^i, r_{t,c}), \alpha^*_{t-1,c-1}(\alpha^i, r_{t,c})\right)$ accordingly, in order to always maintain the optimality of $\left(\alpha^*_{t-1,c}(\alpha^i, r_{t,c}), \alpha^*_{t-1,c-1}(\alpha^i, r_{t,c})\right)$ with respect to $r_{t,c}$ (inner minimization of (8)), but without explicitly solving the minimization problem.

In the following presentation, we use the assumptions that the optimal price $r^*_{t,c}(\alpha)$ increases in α and that (8) is unimodal in $r_{t,c}$ such that the algorithm will find the global optimum. These intuitive assumptions have been formally shown for $C = 1$ in Section 4. As $C > 1$ is not analytically tractable, we verify the first requirement online at runtime and also test the second numerically. In doing so, we never noted any violation. However, if (8) was not unimodal, the algorithm would be stuck in a local optimum. In this case the second algorithm using the discrete action space is preferable.

To determine the price $r_{t,c}$ for the next iteration, we use a simplified version of the optimization problem (8). The construction of this subproblem is based on the following observation: For a sufficiently small increase in $r_{t,c}$, the structure of the minimization's solution, which the greedy procedure determines, does not change. A probability level

$\alpha^*_{t-1,\cdot}$, previously located in a grid point, is still located in this grid point, while the $\alpha^*_{t-1,\cdot}$, located between two grid points, decreases slightly. More precisely, we define the function $\text{int}(\alpha)$ that returns the corresponding interval for a given value of α in the discretization: $\text{int}(\alpha) = j$ if $\alpha^j < \alpha \leq \alpha^{j+1}$. Now the following definition also captures that both $\alpha^*_{t-1,\cdot}$ are located in a grid point: If $s_{t-1,c}^{\text{int}(\alpha^*_{t-1,c})} \leq s_{t-1,c-1}^{\text{int}(\alpha^*_{t-1,c-1})} + r_{t,c}$, then $\alpha^*_{t-1,c}$ is located in a grid point in the minimization's solution and remains in that grid point when $r_{t,c}$ is increased slightly. Otherwise, $\alpha^*_{t-1,c-1}$ remains in a grid point. In both cases, the other $\alpha^*_{t-1,\cdot}$ is decreasing in $r_{t,c}$. The subproblem is now constructed so that it is identical to the original problem (8), but only for sufficiently small increases in $r_{t,c}$.

W.l.o.g., we assume that $\alpha_{t-1,c} \in \mathcal{A}$ remains in the grid point to ease the presentation. This also applies to the flow chart in the next subsection, however, the pseudo-code given there is universal. The subproblem is now obtained by fixing $\alpha^*_{t-1,c}$ and substituting $V^A_{t-1,c-1}$ with an affine linear approximation $V^{sub}_{t-1,c-1}$ that equals $V^A_{t-1,c-1}$ in the interval $\text{int}\left(\alpha^*_{t-1,c-1}\left(\alpha^i, r_{t,c}\right)\right)$; see also Figure 4 (right part) for an illustration of the subproblem corresponding to the minimization's solution. The optimal price r_{opt} in the subproblem can be easily calculated in closed form. If r_{opt} is close enough to the previous price $r_{t,c}$, the subproblem is identical to the original problem (8), and r_{opt} is also optimal for the original problem (8).

5.1.2. Pseudo code and flow chart

More precisely, the algorithm works as follows (see also Algorithm A for pseudo code and Figure 5 for a flow chart). At the beginning of the iteration for α^i, some initializations are performed. Steps 1–5 basically calculate the slopes of $V^A_{t-1,c}$ and $V^A_{t-1,c-1}$ and assign $r_{t,c}$. Step 6 is the core of the algorithm and represents the successive solution of the subproblems described in the previous subsection. Step 6.1 calculates the solution r_{opt} of the subproblem and the two bounds r_{slope} and r_{limit} to check the validity of the current subproblem. We calculate the bounds such that the subproblem equals the original problem for $r_{t,c} \leq \min\{r_{slope}, r_{limit}\}$. If r_{opt} is larger than this threshold, $r_{t,c}$ is set to $\min\{r_{slope}, r_{limit}\}$ and a new subproblem is constructed for the next iteration. The two bounds and r_{opt} are calculated as follows:

Case 1: $s_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})} + r_{t,c} \geq s_{t-1,c}^{\text{int}(\alpha_{t-1,c})}$

$$r_{opt} = \frac{\left(\left(V^A_{t-1,c}\left(\alpha_{t-1,c}^{\text{int}(\alpha_{t-1,c})+1}\right) - V^A_{t-1,c-1}\left(\alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})+1}\right)\right) - \left(\alpha_{t-1,c}^{\text{int}(\alpha_{t-1,c})+1} - \alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})+1}\right) \cdot s_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})} + \alpha_{t,c}\right)}{2\alpha_{t-1,c}^{\text{int}(\alpha_{t-1,c})+1}}$$

$$r_{slope} = s_{t-1,c}^{\text{int}(\alpha_{t-1,c})+1} - s_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})}$$

$$r_{limit} = \left(\alpha^i - \alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})}\right) / \left(\alpha_{t-1,c} - \alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})}\right).$$

Case 2: otherwise,

$$r_{opt} = \frac{\left(\left(V^A_{t-1,c}\left(\alpha_{t-1,c}^{\text{int}(\alpha_{t-1,c})+1}\right) - V^A_{t-1,c-1}\left(\alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})+1}\right)\right) - \left(\alpha_{t-1,c}^{\text{int}(\alpha_{t-1,c})+1} - \alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})+1}\right) \cdot s_{t-1,c}^{\text{int}(\alpha_{t-1,c})} + \alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})+1}\right)}{2\alpha_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})+1}}$$

$$r_{slope} = s_{t-1,c}^{\text{int}(\alpha_{t-1,c})} - s_{t-1,c-1}^{\text{int}(\alpha_{t-1,c-1})}$$

$$r_{limit} = \left(\alpha^i - \alpha_{t-1,c-1}\right) / \left(\alpha_{t-1,c}^{\text{int}(\alpha_{t-1,c})} - \alpha_{t-1,c-1}\right).$$

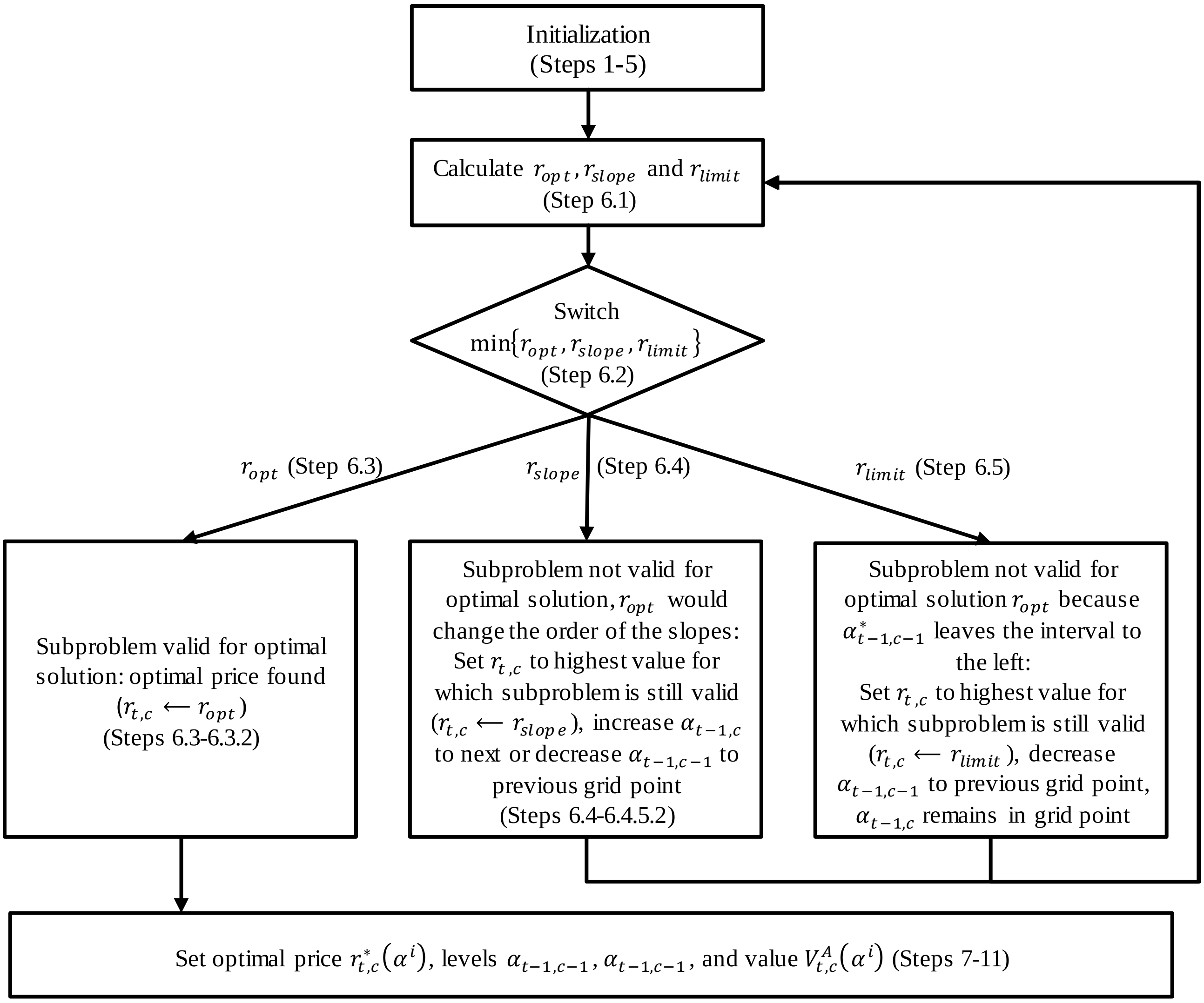

Figure 5: (Simplified) flow chart of Algorithm A

Based on the optimal price and the threshold calculated as shown above, Step 6.2 determines whether and how to construct the next subproblem. Figure 6 illustrates possible cases. Please note that the figure displays α on the horizontal axis and that $\alpha_{t-1,c-1}(r)$ decreases in r. The optimal α from the last iteration are displayed in Figure 6 (a).

Step 6.3 represents $r_{opt} = \min\{r_{opt}, r_{slope}, r_{limit}\}$ and is illustrated in Figure 6 (b). This is the case where the subproblem's solution is optimal for the original problem because the increase in $r_{t,c}$ (and, thus, the decrease in $\alpha_{t-1,c-1}$) is small enough. Thus, the solution of the minimization problem as assumed in the simplified subproblem is also valid for the original minimization problem. We have $r^*_{t,c}(\alpha^i) = r_{opt}$.

Step 6.4 represents $r_{opt} > r_{slope} \wedge r_{slope} \leq r_{limit}$ (see Figure 6 (c, left part)). In this case, the increase of $r_{t,c}$ to r_{opt} would change the order of the slopes $s_{t-1,c}^{\mathrm{int}\left(\alpha^*_{t-1,c-1}(\alpha^i, r_{t,c})\right)} + r_{t,c}$ and $s_{t-1,c}^{\mathrm{int}\left(\alpha^*_{t-1,c}(\alpha^i, r_{t,c})\right)+1}$, so that the fixed $\alpha^*_{t-1,c}$ is no longer in the same grid point. Thus, the price is only increased to $r_{t,c} = r_{slope}$. For this value, the current order of the slopes is still valid. However, as now two slopes have the same value by construction, there is a second valid order that will be used in the next iteration. Finally, $\alpha^*_{t-1,c}(\alpha^i, r_{slope})$ and $\alpha^*_{t-1,c-1}(\alpha^i, r_{slope})$ are determined by increasing $\alpha_{t-1,c}$ to the next grid point, if possible, or decreasing $\alpha_{t-1,c-1}$ to the previous grid point and calculating the other $\alpha_{t-1,\cdot}$, that might be between two grid points, according to (9). This is illustrated in Figure 6 (c, left part), where the circles denote the $\alpha_{t-1,c}$ increased to the next grid point and the $\alpha_{t-1,c-1}$ calculated according to (9). Figure 6 (c, right part) visualizes the next subproblem to consider.

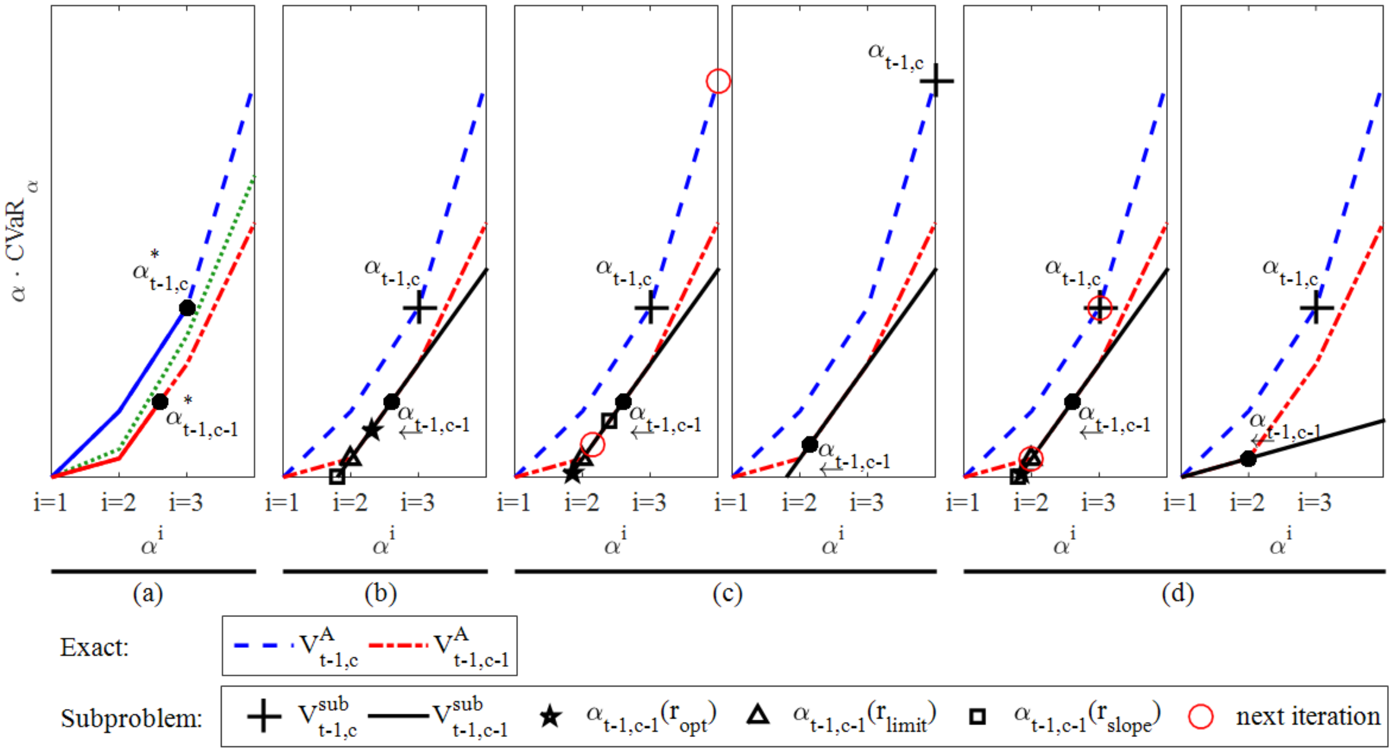

Figure 6: Illustration of the cases arising after the solution of the subproblem in Algorithm A

Step 6.5 represents $r_{opt} > r_{limit} \wedge r_{limit} < r_{slope}$. In this case, the probability level $\alpha^*_{t-1,c-1}$ leaves the interval to the left (i.e. $\alpha^*_{t-1,c-1}(\alpha^i, r_{opt}) < \alpha^{\text{int}(\alpha^*_{t-1,c-1}(\alpha^i, r_{t,c}))}$). This is illustrated in Figure 6 (d, left part). Note that $\alpha^*_{t-1,c-1}$ cannot leave the interval to the right, because $\alpha_{t-1,c-1}$ is non-increasing in the price. The price is increased to $r_{t,c} = r_{limit}$ and $\alpha^*_{t-1,c-1}(\alpha^i, r_{limit})$ has to be set to the previous grid point $\alpha^{\text{int}(\alpha_{t-1,c-1})}$. The level $\alpha^*_{t-1,c}(\alpha^i, r_{limit})$ remains constant and in its grid point. The current subproblem is still valid for this solution. In addition, a second subproblem is valid for this solution (e.g. Figure 6 (d, right part)) and will be considered in the next iteration.

After Steps 6.4 and 6.5, the next subproblem to consider is constructed as outlined above. This procedure is repeated until Step 6.2 applies and r_{opt} from the subproblem is optimal for the original problem.

Steps 7-11 perform some post processing. More precisely, Step 7 sets the optimal price. Steps 8–11 calculate the $\alpha_{t-1,\cdot} \notin \mathcal{A}$. Step 11 calculates the value function.

Algorithm A: Optimization for discrete α

Input: value functions $V^A_{t-1,c}(\cdot)$, $V^A_{t-1,c-1}(\cdot)$, α-grid $\alpha^i \in \{0 = \alpha^1, \ldots, \alpha^{n_\alpha} = 1\}$, number of grid point $i \geq 2$ to be optimized and price $r^*_{t,c}(\alpha^{i-1})$ from last iteration, if available
Output: optimal values for grid point i: $r^*_{t,c}(\alpha^i)$, $V^A_{t,c}(\alpha^i)$

1. Calculate $s^{i-1}_{t-1,c}$, $s^{i-1}_{t-1,c-1}$ ▷ start of initializations: calculate slopes of $V^A_{t-1,c}(\cdot)$ and $V^A_{t-1,c-1}(\cdot)$ in the interval $i-1$
2. **If** $i > 2$ **then** $r_{t,c} = max\{r^*_{t,c}(\alpha^{i-1}), s^{i-1}_{t-1,c} - s^{i-1}_{t-1,c-1}\}$ ▷ initialize price with last price or difference of slopes
3. **Else** $r_{t,c} = s^{i-1}_{t-1,c} - s^{i-1}_{t-1,c-1}$ ▷ price from last iteration not available in first iteration; remember that $\alpha^1 = 0$
4. Calculate $(\alpha_{t-1,c}, \alpha_{t-1,c-1})$ dependent on $(\alpha^i, r_{t,c})$ ▷ e.g. using continuous knapsack

5. $isOptimal \leftarrow false$ ▷ end of initializations
6. **While** not $isOptimal$ ▷ repeat until optimal solution found
6.1. Calculate r_{opt}, r_{slope} and r_{limit}
6.2. **Switch** $min\{r_{opt}, r_{slope}, r_{limit}\}$
6.3. **Case** r_{opt} ▷ subproblem valid for r_{opt}, solution found
6.3.1. **if** $r_{opt} > r_{t,c}$ **then** $r_{t,c} \leftarrow r_{opt}$
6.3.2. $isOptimal \leftarrow true$
6.4. **Case** r_{slope} ▷ subproblem not valid for r_{opt} because order of slopes changes
6.4.1. $r_{t,c} \leftarrow r_{slope}$ ▷ increase $r_{t,c}$ as far as possible, with r_{slope} both old and new order of slopes valid
6.4.2. Calculate $s_{t-1,c}^{int(\alpha_{t-1,c})}, s_{t-1,c-1}^{int(\alpha_{t-1,c-1})}$ ▷ calculate slopes
6.4.3. **If** $s_{t-1,c-1}^{int(\alpha_{t-1,c-1})} + r_{t,c} \geq s_{t-1,c}^{int(\alpha_{t-1,c})}$ **then** $k \leftarrow 1$ ▷set $k = 1$ if $\alpha_{t-1,c}$ in a grid point ($\alpha_{t-1,c} \in \mathcal{A}$) …
6.4.3.1. **Else** $k \leftarrow 0$ … allows $\alpha_{t-1,c}$ to jump to the next grid point
6.4.4. **If** $\alpha^i > r_{t,c} \cdot \alpha^{int(\alpha_{t-1,c})+1+k} + (1 - r_{t,c}) \cdot \alpha^{int(\alpha_{t-1,c-1})}$ ▷ can $\alpha_{t-1,c}$ be increased to next grid point?
6.4.4.1. Increase $\alpha_{t-1,c}$ to the next grid point $\alpha^{int(\alpha_{t-1,c})+1+k}$
6.4.4.2. $\alpha_{t-1,c-1} \leftarrow (\alpha^i - r_{t,c} \cdot \alpha_{t-1,c})/(1 - r_{t,c})$
6.4.5. **Else**
6.4.5.1. Decrease $\alpha_{t-1,c-1}$ to the previous grid point $\alpha^{int(\alpha_{t-1,c-1})}$
6.4.5.2. $\alpha_{t-1,c} \leftarrow (\alpha^i - (1 - r_{t,c}) \cdot \alpha_{t-1,c-1})/r_{t,c}$
6.5. **Case** r_{limit} ▷ subproblem not valid for r_{opt} because r_{opt} is too small
6.5.1. $r_{t,c} \leftarrow r_{limit}$ ▷ increase $r_{t,c}$ as far as possible, for r_{limit} both current and new subproblem valid
6.5.2. Calculate $s_{t-1,c}^{int(\alpha_{t-1,c})}, s_{t-1,c-1}^{int(\alpha_{t-1,c-1})}$ ▷ calculate slopes
6.5.3. **If** $s_{t-1,c-1}^{int(\alpha_{t-1,c-1})} + r_{t,c} \geq s_{t-1,c}^{int(\alpha_{t-1,c})}$ **then** Decrease $\alpha_{t-1,c-1}$ to the previous grid point $\alpha^{int(\alpha_{t-1,c-1})}$
▷ check if $\alpha_{t-1,c}$ is in a grid point (true if $\alpha_{t-1,c} \in \mathcal{A}$)
6.5.4. **Else** Decrease $\alpha_{t-1,c}$ to the previous grid point $\alpha^{int(\alpha_{t-1,c})}$
7. $r_{t,c}^*(\alpha^i) \leftarrow r_{t,c}$ ▷ post processing
8. Calculate $s_{t-1,c}^{int(\alpha_{t-1,c})}, s_{t-1,c-1}^{int(\alpha_{t-1,c-1})}$
9. **If** $s_{t-1,c-1}^{int(\alpha_{t-1,c-1})} + r_{t,c}^*(\alpha^i) \geq s_{t-1,c}^{int(\alpha_{t-1,c})}$ **then** $\alpha_{t-1,c-1} \leftarrow (\alpha^i - r_{t,c}^*(\alpha^i) \cdot \alpha_{t-1,c})/(1 - r_{t,c}^*(\alpha^i))$
10. **Else** $\alpha_{t-1,c} \leftarrow (\alpha^i - (1 - r_{t,c}^*(\alpha^i)) \cdot \alpha_{t-1,c-1})/r_{t,c}^*(\alpha^i)$
11. $V_{t,c}^A(\alpha^i) \leftarrow r_{t,c}^*(\alpha^i) \cdot V_{t-1,c}^A(\alpha_{t-1,c}) + (1 - r_{t,c}^*(\alpha^i)) \cdot [V_{t-1,c-1}^A(\alpha_{t-1,c-1}) + r_{t,c}^*(\alpha^i)]$

5.2 Algorithm B (discrete action space)

5.2.1. Overview

The second possibility to build an efficient algorithm is to discretize the action space. More precisely, we discretize the domain of r, (0,1), by a finite set of n_r points $0 < r^1 < \cdots < r^k < \cdots < r^{n_r} < 1$. This approach can be easily justified, as the prices need to be discretized for all practical purposes. The basic approach to calculate the value function in period t from the value function in period $t-1$ closely follows the structure of (8) with the inner minimization and the outer maximization, and comprises two phases (Figure 7).

To calculate the value function $V_{t,c}^B(\alpha)$, $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha)$ are needed (left part of Figure 7). In the first phase, the minimization is solved for all possible prices $r^k, k = 1, \ldots, n_r$ independently. Note that the value functions $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha)$ are piecewise linear and convex in α. Thus, for a given price r^k, the inner minimization problem is again a continuous knapsack problem the greedy procedure mentioned in Section 5.1 can solve. However, the procedure is not aborted when a given probability level $\alpha_{t,c}$ is reached, but continues until all slopes have been considered and $\alpha_{t-1,c} = \alpha_{t-1,c-1} = 1$. Thus, for each price r^k, we obtain the piecewise linear value function $V_{t,c}^B(\alpha, r^k)$. Figure 7 (middle part) illustrates these value functions for three different prices $r^1 < r^2 < r^3$. Subsequently, the value function $V_{t,c}^B(\alpha)$ can be easily determined as the pointwise maximum of all the value functions $V_{t,c}^B(\alpha, r^k), k = 1, \ldots, n_r$ (right part of Figure 7). Since the $V_{t,c}^B(\alpha, r^k)$ are convex and piecewise linear for all k, $V_{t,c}^B(\alpha)$ inherits these properties.

5.2.2. Pseudo code

After calculating all slopes of $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha)$ in Step 1, the inner minimization is solved in Step 2 of the pseudo code. This part of the algorithm is analogous to the greedy procedure already mentioned in Section 5.1. Therefore, we only describe the most important steps in the following. First, all slopes of $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha) + r^k \cdot \alpha, k = 1, \ldots, n_r$, are sorted as preparatory work. The only difference to Section 5.1 is that now the locations of the sharp points of $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha)$ may differ as the state space is not discretized in Algorithm B. Since all slopes of $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha) + r^k \cdot \alpha, k = 1, \ldots, n_r$, are considered in the minimization, the resulting value functions $V_{t,c}^B(\alpha, r^k), k = 1, \ldots, n_r$, inherit all these slopes. Only the locations of the sharp points will again be different in $V_{t,c}^B(\alpha, r^k)$ as compared to $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha)$. Second, the locations of the sharp points of $V_{t,c}^B(\alpha, r^k)$, denoted by $0 = \alpha_{t,c}^1(r^k), \ldots, \alpha_{t,c}^{n_k}(r^k) = 1$, and the corresponding function values respectively are calculated efficiently by taking whole intervals between two consecutive sharp points of $V_{t-1,c}^B(\alpha)$ and $V_{t-1,c-1}^B(\alpha) + r^k \cdot \alpha$ into account in each iteration. The obtained value functions $V_{t,c}^B(\alpha, r^k)$ are stored efficiently by only memorizing the corresponding point-value pairs $\left(\alpha_{t,c}^1(r^k), V_{t,c}^B(\alpha_{t,c}^1, r^k)\right), \ldots, \left(\alpha_{t,c}^{n_k}(r^k), V_{t,c}^B(\alpha_{t,c}^{n_k}, r^k)\right)$. These pairs correspond to the sharp points in Figure 7 (middle part).

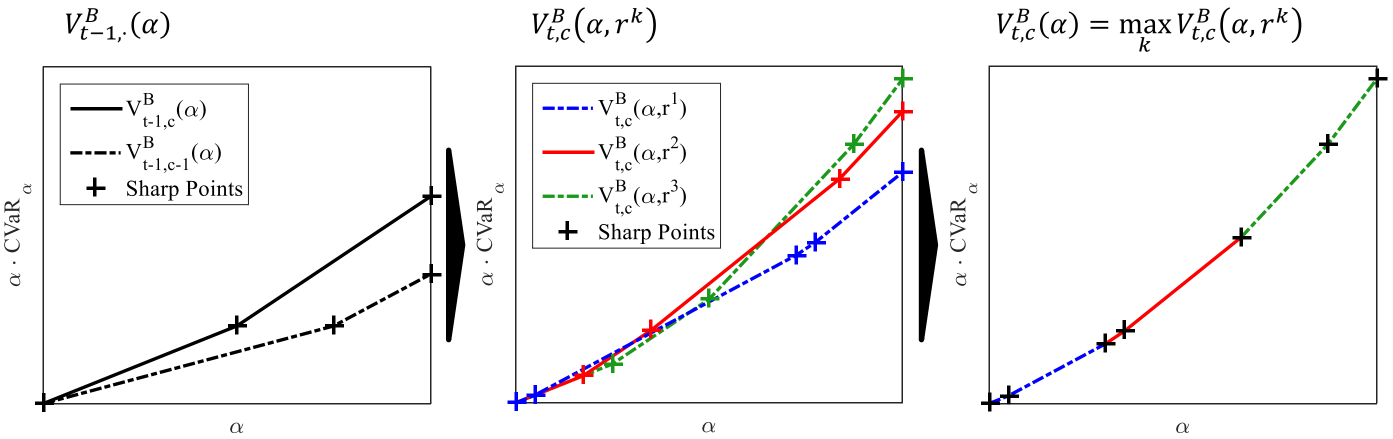

Figure 7: Illustration of Algorithm B

Steps 3 and 4 are just an initialization and a calculation of the expected-value optimal price $r^{l_{t,c}}$ respectively. The outer maximization is considered in Steps 5–7 of the pseudo code. Step 5 is the main part of the determination of the maximal $V_{t,c}^B(\alpha, r^k)$ for every α. To simplify this process, we use the observation that $V_{t,c}^B(\alpha, r^k)$ and $V_{t,c}^B(\alpha, r^{k+1})$ intersect in at most one point and these intersections are ordered. Let $\hat{\alpha}$ and $\tilde{\alpha}$ be the levels of two intersections such that $V_{t,c}^B(\hat{\alpha}, r^{k-1}) = V_{t,c}^B(\hat{\alpha}, r^k)$ and $V_{t,c}^B(\tilde{\alpha}, r^k) = V_{t,c}^B(\tilde{\alpha}, r^{k+1})$. Then, $\hat{\alpha} < \tilde{\alpha}$ and, thus, $V_{t,c}^B(\hat{\alpha}, r^k) < V_{t,c}^B(\tilde{\alpha}, r^k)$. Accordingly, with increasing α the $V_{t,c}^B(\alpha, r^k)$ become maximal in ascending order of the corresponding prices until k reaches the index of the expected-value optimal price $r^{l_{t,c}}$. Thus, Step 5 determines the intersections between

$V_{t,c}^B(\alpha, r^k)$ and $V_{t,c}^B(\alpha, r^{k+1})$ for every $k \leq l_{t,c} - 1$. Note that the function $V_{t,c}^B(\alpha)$ is equal to $V_{t,c}^B(\alpha, r^k)$ until the intersection (Steps 5.2.1.1 – 5.2.1.5; see also Figure 7, middle and right part). Steps 6 and 7 complete the maximization by setting $V_{t,c}^B(\alpha) = V_{t,c}^B(\alpha, r^{l_{t,c}})$ for $\alpha \in \left[\alpha_{t,c}^{j_{l_{t,c}}}(r^{l_{t,c}}), 1\right]$ as the expected-value optimal price is also $CVaR_\alpha$-optimal in this interval.

Algorithm B: Optimization for discrete prices

Input: piecewise linear $V_{t-1,c}^B(\cdot)$, $V_{t-1,c-1}^B(\cdot)$ as pairs $\left(\alpha_{t-1,c-i}(r^k), V_{t-1,c-i}^B(\alpha_{t,c-i})\right)$, price grid $r^1, \dots, r^{n_r}$

Output: optimal prices $r_{t,c}^*(\cdot)$ and values $V_{t,c}^B(\cdot)$ for all sharp points

1. Calculate $s_{t-1,c}^{\cdot}$, $s_{t-1,c-1}^{\cdot}$ ▷ initialization: calculate all slopes of value functions in period $t-1$
2. **For** $k = 1$ ***to*** n_r ▷ for each price point, solve inner minimization of (8) by knapsack to obtain $V_{t,c}^B(\alpha_{t,c}, r^k)$

2.1. $\alpha_{t-1,c-i} \leftarrow 0$, $i = 0,1$, $\alpha_{t,c}^1(r^k) \leftarrow 0$ and $V_{t,c}^B(\alpha_{t,c}^1(r^k), r^k) \leftarrow 0$ ▷ initialization for greedy procedure

2.2. Sort all unique slopes $s_{t-1,c-i}^j + i \cdot r^k$ in ascending order $s_{(1)} < s_{(2)} < \dots < s_{(n-1)}$
▷ sort all slopes of $V_{t-1,c}^B(\cdot)$ and $V_{t-1,c-1}^B(\cdot) + r^k$

2.3. **For** $j_k = 2$ ***to*** n ▷ take all slopes of $V_{t-1,c}^B(\cdot)$ and $V_{t-1,c-1}^B(\cdot) + r^k$ into consideration in ascending order

2.3.1. $S \leftarrow \left\{(i,j): s_{t-1,c-i}^j = s_{(j_k-1)}\right\}$ ▷ select slopes identical to $s_{(j_k-1)}$; at most 2, one from c and one from $c-1$

2.3.2. **For** $(i,j) \in S$ ▷ next sharp point is ‘after’ all identical (see above) slopes

2.3.2.1. $\alpha_{t-1,c-i} \leftarrow \alpha_{t-1,c-i}^{j+1}$ ▷ set both α to the values included in next sharp point

2.3.3. $\alpha_{t,c}^{j_k}(r^k) \leftarrow r^k \cdot \alpha_{t-1,c} + (1 - r^k) \cdot \alpha_{t-1,c-1}$ ▷ calculate location (α) of next sharp point of $V_{t,c}^B(\alpha, r^k)$

2.3.4. $V_{t,c}^B\left(\alpha_{t,c}^{j_k}(r^k), r^k\right) \leftarrow r^k \cdot V_{t-1,c}^B(\alpha_{t-1,c}) + (1 - r^k) \cdot (V_{t-1,c-1}^B(\alpha_{t-1,c-1}) + r^k \cdot \alpha_{t-1,c-1})$
▷ calculate value of $V_{t,c}^B(\alpha, r^k)$ at next sharp point

2.3.5. $s_{t,c}^{j_k-1}(r^k) \leftarrow s_{(j_k-1)}$ ▷ set slope before aforementioned sharp point

3. $j \leftarrow 1$, $\alpha_{t,c}^1 \leftarrow 0$ and $V_{t,c}^B(\alpha_{t,c}^1) \leftarrow 0$ ▷ initialization
4. $l_{t,c} \leftarrow min\left\{argmax_k \, V_{t,c}^B(1, r^k)\right\}$ ▷ calculate expected value optimal price
5. **For** $k = 1$ ***to*** $l_{t,c} - 1$ ▷ determination of the maximal $V_{t,c}^B(\alpha, r^k)$ for every α

5.1. $j_k \leftarrow min\left\{j_k: \alpha_{t,c}^j \leq \alpha_{t,c}^{j_k}(r^k)\right\}$, $j_{k+1} \leftarrow min\left\{j_{k+1}: \alpha_{t,c}^j \leq \alpha_{t,c}^{j_{k+1}}(r^{k+1})\right\}$
▷ calculate the next sharp points of $V_{t,c}^B(\alpha, r^k)$ and $V_{t,c}^B(\alpha, r^{k+1})$ to consider

5.2. **While** $V_{t,c}^B\left(min\left\{\alpha_{t,c}^{j_k}(r^k), \alpha_{t,c}^{j_{k+1}}(r^{k+1})\right\}, r^k\right) \geq V_{t,c}^B\left(min\left\{\alpha_{t,c}^{j_k}(r^k), \alpha_{t,c}^{j_{k+1}}(r^{k+1})\right\}, r^{k+1}\right)$
▷ stepwise increase of sharp points until $V_{t,c}^B(\cdot, r^k)$ is passed by $V_{t,c}^B(\cdot, r^{k+1})$

5.2.1. **If** $\alpha_{t,c}^{j_k}(r^k) \leq \alpha_{t,c}^{j_{k+1}}(r^{k+1})$ **then**
▷ is the sharp point currently considered on $V_{t,c}^B(\alpha, r^k)$ or on $V_{t,c}^B(\alpha, r^{k+1})$ to the left?

5.2.1.1. $j \leftarrow j + 1$
▷ sharp point on $V_{t,c}^B(\alpha, r^k)$ is to the left, $V_{t,c}^B(\alpha)$ still equals $V_{t,c}^B(\alpha, r^k)$; thus, sharp point, optimal price and objective value are stored:

5.2.1.2. $\alpha_{t,c}^j \leftarrow \alpha_{t,c}^{j_k}(r^k)$

5.2.1.3. $r_{t,c}^*(\alpha_{t,c}^j) \leftarrow r^k$

5.2.1.4. $V_{t,c}^B(\alpha_{t,c}^j) \leftarrow V_{t,c}^B(\alpha, r^k)$

5.2.1.5. $j_k \leftarrow j_k + 1$ ▷ move to the next sharp point of $V_{t,c}^B(\cdot, r^k)$

5.2.2. **Else** ▷ sharp point on $V_{t,c}^B(\alpha, r^{k+1})$ is to the left, no consideration for $V_{t,c}^B(\alpha)$

5.2.2.1. $j_{k+1} \leftarrow j_{k+1} + 1$ ▷ move to the next sharp point of $V_{t,c}^B(\cdot, r^{k+1})$
▷ end of while-loop: j_k and j_{k+1} are the sharp points after intersection of $V_{t,c}^B(\cdot, r^k)$ and $V_{t,c}^B(\cdot, r^{k+1})$

5.3. $j \leftarrow j + 1$

5.4. $\alpha_{t,c}^{j} \leftarrow min\left\{\alpha_{t,c}^{j_k}(r^k), \alpha_{t,c}^{j_{k+1}}(r^{k+1})\right\} - \frac{V_{t,c}^{B}\left(min\left\{\alpha_{t,c}^{j_k}(r^k),\alpha_{t,c}^{j_{k+1}}(r^{k+1})\right\},r^{k+1}\right)-V_{t,c}^{B}\left(min\left\{\alpha_{t,c}^{j_k}(r^k),\alpha_{t,c}^{j_{k+1}}(r^{k+1})\right\},r^k\right)}{s_{t,c}^{j_{k+1}-1}(r^{k+1})-s_{t,c}^{j_k-1}(r^k)}$

▷ calculate the exact point of intersection to store it as next sharp point of $V_{t,c}^{B}(\cdot)$; also store the optimal price and the objective value:

5.5. $r_{t,c}^{*}\left(\alpha_{t,c}^{j}\right) \leftarrow r^k$

5.6. $V_{t,c}^{B}\left(\alpha_{t,c}^{j}\right) \leftarrow V_{t,c}^{B}\left(\alpha_{t,c}^{j}, r^k\right)$

$j_{l_{t,c}} \leftarrow min\left\{j_{l_{t,c}}: \alpha_{t,c}^{j} \leq \alpha_{t,c}^{j_{l_{t,c}}}(r^{l_{t,c}})\right\}$ ▷ interval in which the expected-value optimal price is also $CVaR_\alpha$-optimal

6. **For** $j_l = j_{l_{t,c}}$ **to** N

▷set $V_{t,c}^{B}(\alpha) = V_{t,c}^{B}\left(\alpha, r^{l_{t,c}}\right)$ and $r_{t,c}^{*}(\alpha)$ to the expected-value optimal price for $\alpha \in \left[\alpha_{t,c}^{j_{l_{t,c}}}\left(r^{l_{t,c}}\right), 1\right]$

6.1. $j \leftarrow j + 1$

6.2. $\alpha_{t,c}^{j} \leftarrow \alpha_{t,c}^{j_l}\left(r^{l_{t,c}}\right)$

6.3. $r_{t,c}^{*}\left(\alpha_{t,c}^{j}\right) \leftarrow r^k$

6.4. $V_{t,c}^{B}\left(\alpha_{t,c}^{j}\right) \leftarrow V_{t,c}^{B}\left(\alpha_{t,c}^{j_l}\left(r^{l_{t,c}}\right), r^{l_{t,c}}\right)$

6 Numerical examples

The following pricing mechanisms were implemented to evaluate the effectiveness of the proposed approaches:

- *A* is a mechanism that uses Algorithm A (Section 5.1) to approximately solve (8) and calculate the CVaR_α-optimal dynamic price. Wherever necessary, we indicate the grid size $i \in \{0.1, 0.01, 0.001, 0.0001\}$ used for the discretization of the probability level α by writing A_i.
- *B* is a mechanism that uses Algorithm B (Section 5.2) to approximately solve (8) and calculate the CVaR_α-optimal dynamic price. Again, we indicate the discretization $i \in \{0.1, 0.01\}$ used for the price r by writing B_i. Owing to high runtimes, we did not consider finer grids for *B*.
- *EV-Dyn* is the classical dynamic pricing mechanism that maximizes the expected value (Section 3.2).
- *CVaR-Fix* is another benchmark mechanism that determines the CVaR_α-optimal fixed price. Customers' willingness-to-pay is i.i.d. and uniformly distributed on $[0,1]$. Thus, the total number of customers willing to buy an item at a fixed price r_{FP} is binomially distributed with parameter T (total number of customers arriving during the booking horizon from period T to period 1) and success probability $1 - r_{FP}$. Considering the initial capacity, no more than C items can be sold. Let $N(r_{FP}) \in \{0, \dots, C\}$ denote the random variable indicating the total number of items sold at fixed price r_{FP}. Then, the total revenue equals $r_{FP} \cdot N(r_{FP})$. Determining the optimal fixed price $r_{FP}^{*}(\alpha) = \arg\max_{r_{FP} \in [0,1]} CVaR_\alpha\left(r_{FP} \cdot N(r_{FP})\right)$ is numerically quite easy, but generally it is not possible to derive solutions in closed form.
- *EV-Fix* is a variant of *CVaR-Fix* with $\alpha = 1$ and determines the expected-value optimal fixed price.
- *Exact* is the solution of (8) given by Proposition 1 and only applicable to $C = 1$.

The main output of the aforementioned mechanisms are policies with (approximately) optimal selling prices r. In addition, they also compute a corresponding value function $\tilde{V}(\alpha)$ for each level α. However, this value function is only an approximation with mechanisms *A* and *B*, because of their inherent discretization. The value function only equals the exact CVaR_α for *CVaR-Fix* and *Exact*. *EV-Fix* and *EV-Dyn* are only available for $\alpha = 1$. Thus, the value function is not suitable for evaluating the mechanisms, and therefore simulations are performed. For each setting, we generated 10,000 customer streams in advance to compare the mechanisms by using the same streams. A simulation run corresponds to a complete sales process with a selling price set in each period according to the mechanism investigated and observing the arriving customer's decision before moving on to the next period. Finally, the CVaR_α is calculated over all 10,000 customer streams and denoted by $\tilde{V}^{\mathrm{sim}}(\alpha)$.

The experiments were implemented with MATLAB version R2013a and run on a PC with a 2.8 GHz Intel Core i7 processor and 8 GB of RAM, using Microsoft Windows Server 2008 R2 64 bit. Unless otherwise stated, we report values averaged over levels $\alpha = 0.01, 0.02, \dots, 1$ in this section, and choose $T = 10$ and $C = 1, \dots, 10$, yielding 10 different settings, as we can illustrate various effects more clearly in a small setting. Later, we also show that the mechanisms are still applicable in a bigger setting. As in Section 5, we assume that customers' willingness-to-pay is uniformly distributed on $[0,1]$, i.e. $p(r) = 1 - r$ for $r \in [0,1]$.

In the following, we first compare the mechanisms optimizing CVaR with dynamic prices (*A* and *B*). As we will see in Section 6.1 that mechanism *A* is superior, we focus on A in the following experiments. First, we illustrate the CVaR-optimal prices over time (Section 6.2) and discuss the trade-off between risk-aversion and expected revenue maximization (Section 6.3). We then take a broader perspective and compare mechanism $A_{0.0001}$ to the benchmark mechanisms in Section 6.4. In Section 6.5, we analyze how the resulting prices and CVaRs attained depend on the level α, while Section 6.6 illustrates the capacity's influence.

6.1 Mechanisms *A* and *B* with different grid sizes

Table 1: Runtimes [s] for $C \leq 10, T = 10$

Grid	**0.1**		**0.01**		**0.001**	**0.0001**	***Exact***
T=10	***A***	***B***	***A***	***B***	***A***	***A***	
C=1	0.02	0.11	0.05	3.73	0.94	22.03	0.02
C=5	0.03	0.72	0.22	86.75	2.50	64.32	
C=10	0.02	0.36	0.11	101.82	0.78	21.14	

Table 1 shows the runtimes (in seconds) of mechanisms *A* and *B* for the settings $C \in \{1, 5, 10\}$. The given values only reflect the time necessary to compute the policy and do not include simulations. There are considerable differences between the two approaches. Mechanism *A* is much faster than *B*, with the differences in runtime increasing for discretization with finer grids. In addition, mechanism *A* scales much better with regard to capacity. *Exact* only needs a fraction of the other mechanisms' runtime, but is available only for $C = 1$. The runtimes of mechanisms *A* and *B* for much larger settings are depicted in Figure 8. More precisely, we started with a setting with $C = 3, T = 10$ and proportionally scaled it up until $C = 300, T = 1000$. Therefore, the total number of nontrivial states with $C > 0$ and $T > 0$ is scaled up from 30 to 300,000, upon which the scaling of the horizontal axis of Figure 8 is based. The figure shows that especially mechanism *A* with a not too fine grid size is still applicable for large settings.

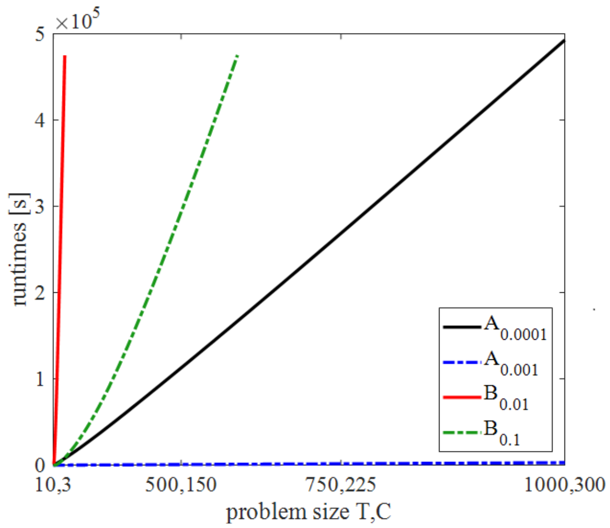

Figure 8: Runtimes for larger settings

We now consider the CVaR $\tilde{V}^{\text{sim}}(\alpha)$ attained when applying the mechanism to a sales process, i.e. the value of the policy. Thereby, we examine how the attained CVaR values behave for different grid sizes, and which mechanism provides the highest CVaR. To this end, we compute the mean relative CVaR (MRC), i.e. the CVaR attained by using a mechanism $\tilde{V}^{\text{sim}}(\alpha)$ relative to the base mechanism $\tilde{V}^{\text{sim}}_{A_{0.0001}}$ which performed superior in pretests:

$$MRC\left(\tilde{V}^{\text{sim}}, \tilde{V}^{\text{sim}}_{A_{0.0001}}\right) = \text{mean}\left(\frac{\tilde{V}^{\text{sim}}(\alpha)}{\tilde{V}^{\text{sim}}_{A_{0.0001}}(\alpha)}\right)$$

where the mean is again over all $\alpha \in \{0.01, 0.02, \dots, 1\}$.

The results are shown in Table 2. As expected, $MRC\left(\tilde{V}^{\text{sim}}, \tilde{V}^{\text{sim}}_{A_{0.0001}}\right)$ increases with finer grid sizes in respect of both mechanisms. For $C \geq 4$ (not all C are shown to save space), $B_{0.1}$ considerably outperforms $A_{0.1}$, but for a grid size of 0.01, the difference is mostly approximately 1%. Mechanism $A_{0.001}$ performs comparable to $B_{0.01}$, and $A_{0.0001}$ consistently delivers a higher CVaR than $B_{0.01}$. Given CVaR's monotonicity in the grid size, it seems likely to prefer even finer grid sizes. However, we think that both the negligible difference between $A_{0.001}$ and $A_{0.0001}$, as well as the equivalence of the CVaR of $A_{0.0001}$ and *Exact* for $C = 1$, are strong signs that a finer grid size will only increase the runtimes.

All in all, mechanism *A* operating on a discretization of the state space is notably faster and yields a better policy than mechanism *B*, which uses a discretization of the action space. Thus, we only consider mechanism *A* in the following subsections.

Table 2: $MRC\left(\tilde{V}^{\text{sim}}, \tilde{V}^{\text{sim}}_{A_{0.0001}}\right)$ – Mean relative CVaR

Grid	**0.1**		**0.01**		**0.001**	**0.0001**	***Exact***
T=10	***A***	***B***	***A***	***B***	***A***	***A***	
C=1	93.86%	99.06%	99.83%	99.92%	99.97%	100.00%	100.00%
C=5	91.80%	98.63%	97.95%	99.83%	99.67%	100.00%	
C=10	91.99%	98.94%	98.23%	99.80%	99.60%	100.00%	

However, there is one problem characteristic that may render mechanism *B* advantageous. If only a few exogenously given price points can be used, mechanism B choses from this set. By contrast, the prices calculated by mechanism *A* have to be rounded to these price points afterwards. Obviously, there are two effects that both are in favor of mechanism B as the cardinality of the set of feasible prices decreases. First, obviously, the smaller the set of prices, the faster mechanism *B* becomes (see also Figure 8). Second, the solution quality of *B* relative to *A* improves because the solution quality of *A* decreases as rounding errors increase. To illustrate this, we calculated both mechanisms using a few equidistant price points between the minimum and maximum willingness-to-pay. Table 3 shows the mean relative CVaR obtained by applying *B* directly relative to rounding A's policy ($\tilde{V}^{\text{sim}}_{A_{0.0001}(B_{Grid})}$). Mechanism *B* can indeed obtain better results than *A* if less than 11 price points are used. If only four price points are used, the advantage of *B* reaches almost 2%.

Table 3: $MRC\left(\tilde{V}^{\text{sim}}, \tilde{V}^{\text{sim}}_{A_{0.0001}(B_{Grid})}\right)$ – Mean relative CVaR

Grid	**0.33**	**0.25**	**0.2**	**0.1**
T=10	***B***	***B***	***B***	***B***
C=1	101.50%	100.45%	100.51%	100.02%
C=5	101.90%	101.52%	101.25%	100.76%
C=10	101.47%	102.16%	101.43%	101.18%

6.2 Illustration of optimal price over time

Our numerical investigation revealed that a fixed price policy is not optimal in the beginning of the selling horizon for capacities larger than one unit, which is contrary to our results for one unit of capacity. Figure 9 shows the optimal prices set in each time period for exemplary streams of customers, with an initial capacity of $C = 2$ and the initial probability levels $\alpha_{10,2} = 0.3$ (left part) and $\alpha_{10,2} = 0.7$ (right part). The risk-averse optimal price is initially $r_{10,2} = 0.62$ (left part) and $r_{10,2} = 0.71$ (right part) and declines over time if no sale occurs. This is analogous to the risk-neutral price, that declines as long as capacity is scarce (displayed only for $C = 1$). If a sale occurs, the price jumps upwards. Apparently, the second selling price is set such that the sum of the revenues obtained from selling the two units of capacity is equal (1.25 in the left part and 1.44 in the right part), no matter when (and at what price) the first sale takes place. The sum depends only on the initial level of $\alpha_{10,2}$. That is, if the first sale happens earlier (e.g. at $t = 9$) and, thus, at a higher price, the second price (for the last unit) is smaller compared to when the first sale happens later (e.g. at $t = 7$) and at a lower price. However, the risk-averse price is still bounded by the risk-neutral price. Thus, the second price cannot exceed the risk-neutral price. If the first sale happens very late (e.g. at $t = 5$ in the right part of Figure 9) and at a very low price, the second price can only be set to the risk-neutral price and the sum may be lower (only 1.40 in this example).

This behavior can be observed for arbitrary capacities and explained as follows. Consider an optimal policy with respective VaR^*_α and CVaR^*_α. The sum of the prices set in the evolution of the sales process is always constant for a given initial α as long as $\alpha_{t,c} < 1$ (with the constant depending on α). This behavior can be derived from the solution of the inner minimization in equation (8). Setting prices which cumulative revenue exceeds VaR^*_α is clearly not optimal, since this event is not taken into account when calculating CVaR^*_α and it also has a lower sales probability than a policy with cumulative revenue smaller than or equal to VaR^*_α. Therefore, one should strive to achieve the highest cumulative revenue taken into consideration, which is VaR^*_α. However, a cumulative revenue of VaR^*_α is not always achievable. In this case, the event is completely captured by CVaR's expectation. Thus, from some time period t onwards, the expected value of the cumulative revenue is maximized and, therefore, $\alpha_{t',c} = 1\ \forall t' \leq t$ and the optimal risk-neutral price is set from time period t until the end of the time horizon.

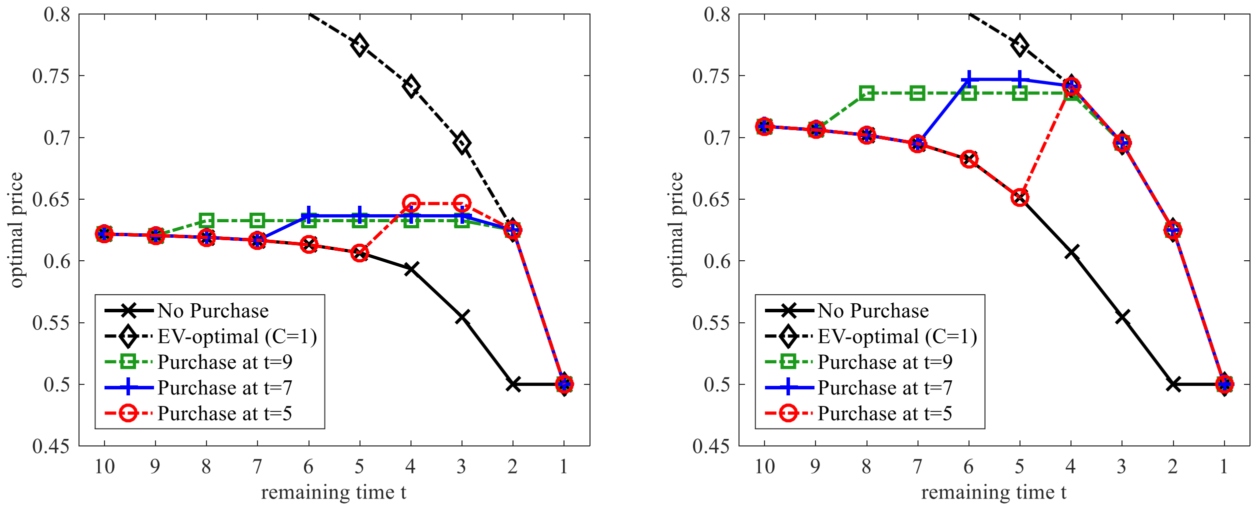

Figure 9: Evolutions of the selling process for 3 different customer streams with α=0.3 (left) and α=0.7 (right)

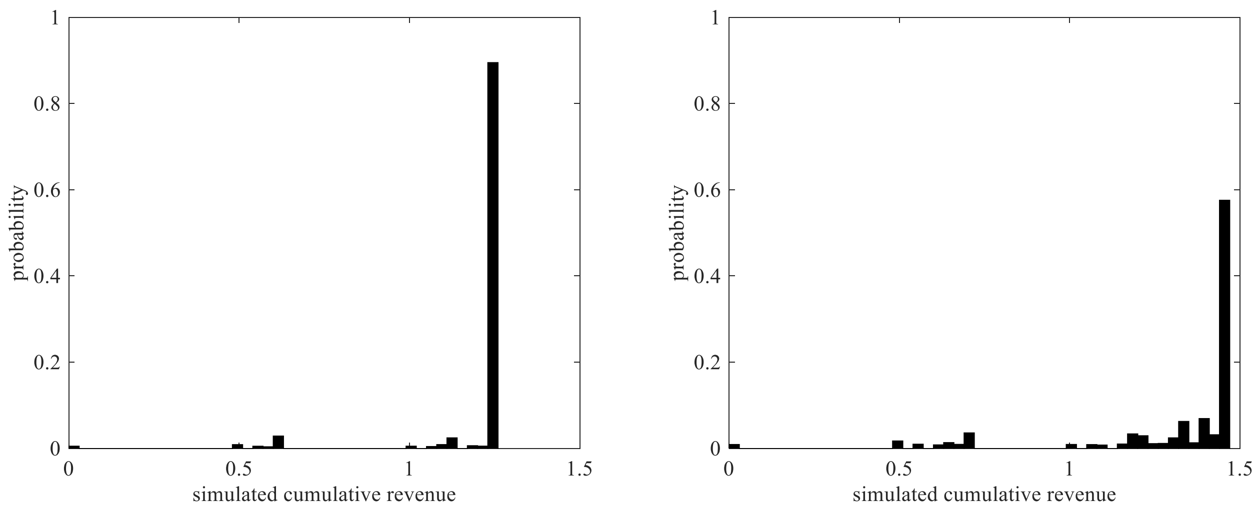

Figure 10: Histogram of total revenues for C=2 T=10 with α=0.3 (left) and α=0.7 (right)

Figure 10 shows the empirical distribution of total revenue obtained with $C = 2$ and $T = 10$. Obviously, total revenue equals the VaR_α with a very high probability as in many scenarios, exactly the VaR_α is obtained (rightmost bar in each histogram). Values higher than VaR_α never occur. Values lower than VaR_α are rarely observed and occur in three clusters. The leftmost cluster contains revenues equal to 0 and captures evolutions without a single sale. In the middle cluster, only one sale happens. In the rightmost cluster, which is next to VaR_α, all two capacity units are sold, but the sales happen so late that the VaR_α is not obtained.

6.3 Risk-aversion vs. expected revenue trade-off

Maximizing CVaR instead of expected revenue obviously reduces the expected value. In this subsection, we illustrate this trade-off between maximizing expected revenue and CVaR. Figure 11 shows data obtained for $T = 10$ time periods to go, $C \in \{1, 4, 7, 10\}$ and $\alpha_{C,10} \in [0.05,1]$. For each combination of α and C, we calculated two policies: the expected value (EV) optimal policy and the CVaR_α-optimal policy obtained from mechanism A. We then evaluated both policies by simulation and recorded expected value and $CVaR_\alpha$-obtained with each policy. Finally, the Gain in CVaR is the ratio of the CVaR_α obtained. The Loss in EV is the ratio of the expected revenues:

$$Gain\ in\ CVaR = \frac{\tilde{V}^{\text{sim}}_{A_{0.0001}}(\alpha) - \tilde{V}^{\text{sim}}_{A_{EV-Dyn}}(\alpha)}{\tilde{V}^{\text{sim}}_{A_{EV-Dyn}}(\alpha)} \text{ and } Loss\ in\ EV = \frac{\tilde{V}^{\text{sim}}_{A_{EV-Dyn}}(1) - \tilde{V}^{\text{sim}}_{A_{0.0001}}(1)}{\tilde{V}^{\text{sim}}_{A_{EV-Dyn}}(1)}$$

Obviously, for $\alpha = 1$ both mechanisms are identical and the gain in CVaR as well as the loss in EV are 0. As α increases, the mechanisms increasingly differ and the gain in CVaR as well as the loss in EV increase. When capacity is ample ($C = 7$ or $C = 10$), the loss in EV is much higher than the gain in CVaR (e.g. $\alpha = 0.5$: loss in EV of about 11% vs. gain in CVaR of 4%, $\alpha = 0.25$: EV -22% vs. CVaR +10%). When capacity is scarce, the gain in CVaR is much higher and, at the same time, the corresponding loss in EV is much lower. For $C = 1$, we obtain EV -3% and CVaR +7% at $\alpha = 0.5$ and EV -8% and CVaR +27% at $\alpha = 0.25$. Thus, the cost in expected value is comparably lower when capacity is scarce.

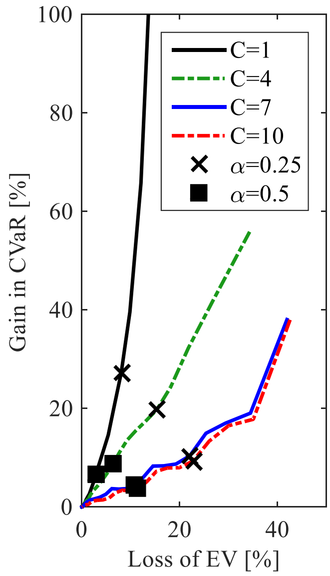

Figure 11: Gain in CVaR vs. loss in expected value through optimization of CVaR for T=10

6.4 Comparison of mechanism A and the benchmark approaches

The $MRC(\tilde{V}^{\text{sim}}, \tilde{V}^{\text{sim}}_{A_{0.0001}})$ of all the other mechanisms relative to $A_{0.0001}$ is shown in Table 4. A distinct order is visible between these three control mechanisms. After $A_{0.0001}$, *CVaR-Fix* performs best for all the capacities, followed by *EV-Dyn* and *EV-Fix*. Although it is a strong restriction to require a static policy, *CVaR-Fix* works surprisingly well in all cases, whereas *EV-Dyn* and *EV-Fix* perform better for higher capacities. Capacity is not scarce for $C = 10$ and, therefore, the optimal price that *EV-Dyn* calculates does not change over time, which leads to the same results for *EV-Dyn* and *EV-Fix* (see, e.g., Dong et al. 2009). Note that the MRC of *CVaR-Fix* is less than 100%, even for $C = 1$, due to the optimal policy only initially being a constant price policy for $C = 1$ and switching to a dynamic one at the end of the booking horizon.

Besides these three approaches, we considered a popular and easy to calculate approach from literature (see Online Supplement S.11 for details). More precisely, we compared mechanism A with the policy obtained from the exponential utility function with constant absolute risk aversion (CARA, $-e^{-\gamma R}$) with parameter γ (see e.g. Lim and Shanthikumar 2007). As there is no one-to-one mapping between α and γ, we tested a broad range of γ-values to find the one with the highest CVaR_α for a given α. We repeated this procedure for several states (c, t) and obtained the following results: CVaR obtained by CARA is considerably lower and the γ-value with the highest CVaR_α depends not only on α but also on the current state (c, t) and the problem parameters. This makes it impossible to choose the best γ a priori.

Table 4: $MRC(\tilde{V}^{\mathrm{sim}}, \tilde{V}^{\mathrm{sim}}_{A_{0.0001}})$ – Mean CVaR of benchmarks relative to $A_{0.0001}$

T=10	*EV-Dyn*	*CVaR-Fix*	*EV-Fix*	*Exact*
C=1	82.91%	98.29%	79.14%	100.00%
C=5	90.01%	99.11%	87.23%	
C=10	93.15%	98.29%	93.15%	

Additional numerical experiments (not shown here) show that the consideration of risk aversion is still relevant in bigger settings, because the probability level changes over time during the selling horizon, but the difference between the dynamic policies of *A* and *EV-Dyn* decreases. The consideration of risk aversion is less relevant when there is more time available to sell the given inventory. Naturally, the static policies perform worse than their dynamic counterparts. But *CVaR-Fix* performs very well in all settings with a stable $MRC(\tilde{V}^{sim}_{CVaR\text{-}Fix}, \tilde{V}^{\mathrm{sim}}_{A_{0.0001}})$. Therefore, a fixed price policy can be a simple and, nonetheless, good choice when considering risk aversion in dynamic pricing. Moreover, if time and capacity are simultaneously scaled up, the consideration of risk-aversion and dynamic pricing becomes less important.

6.5 Illustration of optimal price and CVaR as a function of the probability level

Figure 12 shows the optimal prices calculated by $A_{0.0001}$ and *CVaR-Fix* in $T = 10$ for $C \in \{1,4,7,10\}$ as functions of α on the left-hand side and, on the right-hand side, the CVaRs $\tilde{V}^{\mathrm{sim}}$ attained by all the control mechanisms. We did not plot the optimal prices for *EV-Dyn* and *EV-Fix*, because they correspond to the optimal prices of Algorithm *A* and *CVaR-Fix* for $\alpha = 1$. Moreover, we did not include *Exact* for $C = 1$, because the results were practically identical to $A_{0.0001}$. In addition to that, *EV-Dyn* and *EV-Fix* yield identical prices and CVaRs for $C = 10$.

Regarding the optimal price, in analogy to Proposition 2, we note that the price of mechanism *A* is strictly monotonically increasing in α. Surprisingly, the optimal price of *CVaR-Fix* as a function of α exhibits a saw-tooth pattern for $C = 7$ and $C = 10$. Between intervals, where the optimal price of *CVaR-Fix* increases in α, there are downward price jumps. A detailed analysis (not shown here) shows that the amount of capacity taken into account for the calculation of CVaR_α for different probability levels can explain this behavior. For a given α, only the sale of a specific amount of capacity is considered. The higher α, the more capacity is considered for sale and there is a downward price jump when one more unit of capacity is taken into account.

Regarding the CVaR obtained, we observe that, for low values of α, there are two groups with more or less identical CVaRs: CVaR-maximizing approaches (*A*, *CVaR-Fix*) outperform expected-value maximizing approaches (*EV-Dyn*, *EV-Fix*). The considerable gap between these two groups is relatively bigger for lower values of C. For high levels of α, the consideration of risk aversion is less important and we observe two different groups: approaches with dynamic policies (*A*, *EV-Dyn*) slightly outperform approaches with fixed prices (*CVaR-Fix*, *EV-Fix*). For $\alpha = 1$, the approaches in each group are even formally equivalent. Again, the difference between the groups decreases in C. They are even equivalent for $C = 10$, where capacity is not scarce and dynamic policies also yield a fixed price. Thus, we conclude that for high risk aversion (low α), the consideration of risk aversion seems more important than dynamic pricing, and for low risk aversion (high α), dynamic pricing is more important.

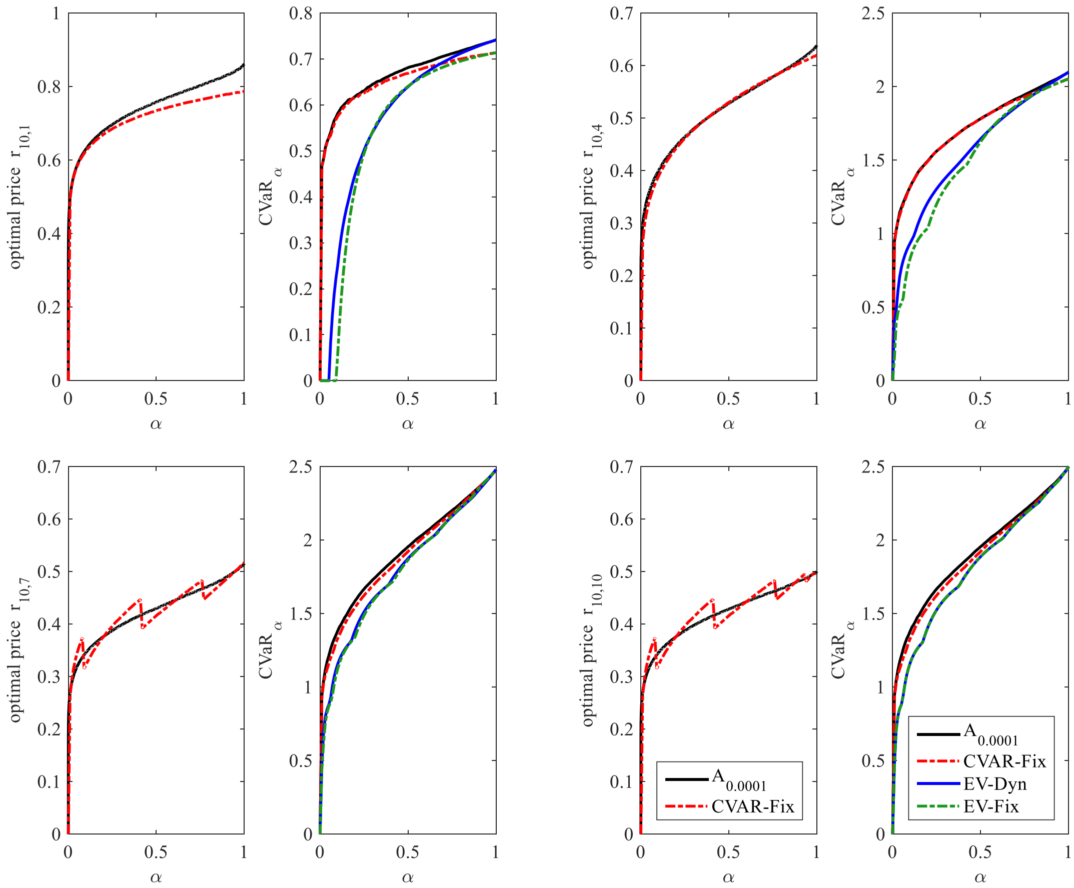

Figure 12: Optimal price and CVaR as a function of α for T=10 and C=1 (upper left), C=4 (upper right), C=7 (lower left), C=10 (lower right)

6.6 Illustration of optimal price and CVaR as a function of capacity

In this section, we repeat the previous analysis, but focus on varying capacity for $T = 10$ and $\alpha \in \{0.3, 0.7\}$, chosen to represent high and low risk aversion. The optimal price and CVaR ($\tilde{V}^{\text{sim}}$) are shown as functions of capacity for all mechanisms in Figure 13.

As already mentioned in Section 0, *CVaR-Fix* takes a specific amount of capacity into account for a probability level. The optimal price and CVaR of *CVaR-Fix* do not change when capacity is higher than this threshold ($C = 4$ in the figure for $\alpha = 0.05$). Thus, in this case, higher capacity has no marginal benefit. Moreover, this threshold increases in the probability level ($C = 5$ in the figure for $\alpha \geq 0.1$) until, in the expected value optimal case of $\alpha = 1$, all units of capacity are taken into account for sale (not shown here). For mechanism *A*, *CVaR-Fix*, and *EV-Fix*, the optimal price is strictly monotonically decreasing in capacity until it remains constant, whereas the optimal price of *EV-Dyn* is strictly monotonically decreasing in the complete interval, following the well-known results for risk-neutral dynamic pricing. The CVaR values ($\tilde{V}^{\text{sim}}$) exhibit the same behavior, but are increasing, of course.

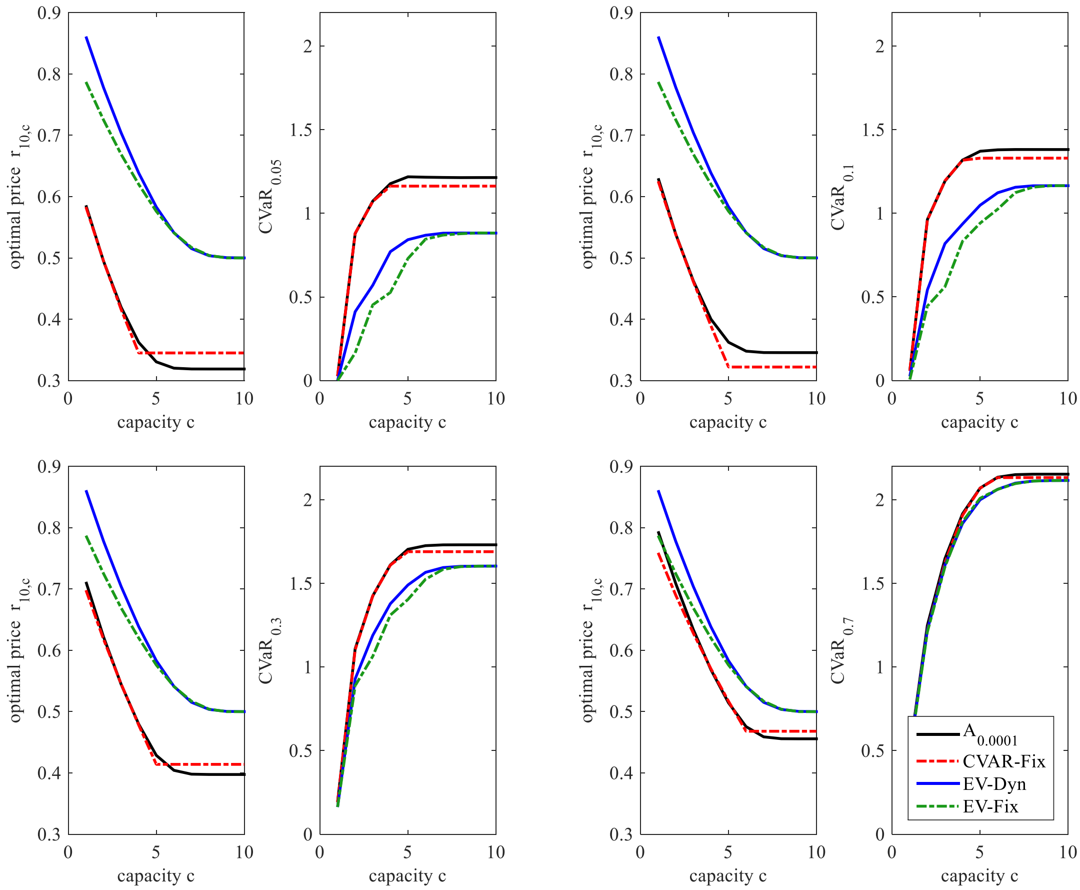

Figure 13: Optimal price and simulated CVaR of several approaches as a function of capacity for T=10, α=0.05 (upper left), α=0.1 (upper right), α=0.3 (lower left), and α=0.7 (lower right)

7 Conclusions

The consideration of risk aversion in dynamic pricing is quite recent. We added to this growing stream of literature by showing how the standard model of dynamic pricing can be generalized to include Conditional Value-at-Risk (CVaR)—a risk measure widely used in other areas due to its desirable theoretical properties and intuitive appeal. However, the time-consistent maximization of CVaR over the whole selling horizon requires the inclusion of CVaR's probability level in the state space. This reflects the decision maker's adaption of risk preferences when new information becomes available over time. For one capacity unit, we analytically solved the model and characterized the optimal policy based on some mild assumptions. An arbitrary capacity was tackled numerically. We developed two algorithms. The first algorithm is based on a discretization of the state space (i.e. the probability level). Its efficiency stems from replacing the complex bilevel problem in each stage with a series of well-chosen subproblems which are solved analytically. The second algorithm considers the bilevel problem's maximization and minimization subsequently, and is based on a discretization of the action space (i.e. the price).

7.1 Managerial implications

The most important managerial implication is that for one unit of capacity, the risk-averse optimal price is constant over large parts of the selling horizon. This contrasts the standard setting of risk-neutral dynamic pricing, where the price continuously declines over time. Whereas the standard recommendation until now was to continuously lower the price, we now recommend a risk-averse seller to start with a lower price and initially keep it constant which is also easier to implement. Interestingly, our recommendation is in line with the price-setting behavior often observed in practice, but often considered irrational. Moreover, depending on the level of risk aversion, every initial price between 0 and the risk-neutral price can be optimal.

For arbitrary capacity, we saw that the cost of risk-aversion is lower with smaller capacities. Moreover, the consideration of risk is more important than the price's dynamic adjustment for most levels of risk aversion. The investigation of bigger settings showed the expected results: Risk aversion is still relevant, although the differences in the CVaR between the approaches diminish with increasing capacity or the length of the selling horizon.

7.2 Extensions

We did not consider a list of some extensions to focus on the essentials of our model and to ease notation. However, it appears quite straightforward to generalize the considerations regarding the following aspects.

- We assumed that a customer arrives for sure in every period and that the willingness-to-pay is time-homogeneous. Nonetheless, a different (nonhomogeneous) arrival probability could be easily integrated into the model. It is also possible to consider a nonhomogeneous (i.e. time-dependent) willingness-to-pay with only minor modifications to the algorithms. However, a few results will obviously change; especially prices might increase over time without sales.
- Furthermore, the incorporation of multi-unit demand is straightforward. The main modification is that in the model presented, two events are distinguished in each period: selling one unit and no sale. With multi-unit demand, additional events corresponding to two and more units sold have to be considered. The extension of Algorithm B is quite straightforward as only the first phase is modified to consider these additional events. The extension of Algorithm A is more cumbersome, especially if nonlinear prices are considered, because many more cases can arise after the solution of the subproblem.
- Regarding discrete prices, properties P.1-P.3 (or their discrete counterparts) are not really necessary for Algorithm B. More precisely, we used them to reduce the computational burden, but a modification checking all intersections of value functions for different prices and not assuming any order would do without the properties. Property P.4 is an adaptation of the standard assumption regarding the presence of a null price in dynamic pricing for the specific setting considered in this paper (see, e.g., Gallego and van Ryzin 1994, Karlin and Carr 1962).

7.3 Future Research

There are several possible avenues for future research. First and foremost, the generalization of the results on $C = 1$ and especially Section 4.2 to $C > 1$ is desirable. Although a few minor properties are easy to show, a very tedious proof by induction along the lines for $C = 1$ but considering many more cases and solutions at borders seems necessary.

Second, from a broader perspective, there is still no consensus in the literature on the definition of time consistency in a dynamic setting. While we follow Pflug and Pichler (2016) and recursively maximize the CVaR over the entire horizon, there is an ongoing discussion (see, e.g., Bamberg and Krapp 2015 and the references therein). For example, Rudloff et al. (2014) prefer a nested calculation with constant probability level and try to give some interpretation. Apart from work on time-consistency itself, this suggests to also consider and compare the optimization of CVaR with a constant level.

Third, we think that especially the result of the constant prices for one unit of capacity is relevant beyond the literature on dynamic pricing. In many markets, sellers with a single item initially set constant prices and only lower the price later on. This downward adjustment is often followed by a quick sale. To date, this behavior has often been explained with price adjustment costs and/or low seller motivation in empirical studies (see, e.g., Knight 2002 and Anglin et al. 2003 for the real estate market). Our results provide a new explanation. We have shown that this behavior is perfectly rational for a risk-averse seller with only one unit of capacity, even if price adjustments are possible. Thus, future empirical studies should take these results into account.

Finally, research on dynamic pricing with demand learning is very active (see the survey by den Boer 2015). However, to the best of our knowledge it has never been investigated how risk-aversion influences the trade-off between exploration of demand and exploitation of the current knowledge.

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Supplement: Optimizing CVaR in Dynamic Pricing

S.1 Properties of the value function

In this section, properties of the value function are stated. In Sections S.2–S.8, we assume these properties hold for $t-1$ and show Propositions 1 – 6 for period t. In Section S.9, we use these propositions and show by induction that the properties hold.

A.1. $0 = V_{t-1,1}(0) < V_{t-1,1}(1) < 1$

A.2. $V'_{t-1,1}(\alpha_{t-1,1})$ exists, is continuous and positive for $\alpha_{t-1,1} > 0$.

A.3. Except a finite number of sharp points of $V'_{t-1,1}(\alpha)$, $V_{t-1,1}(\alpha_{t-1,1})$ is two times differentiable for all $\alpha_{t-1,1} \in (0,1)$ and it holds that $V''_{t-1,1}(\alpha_{t-1,1}) > 0$. At these sharp points, the subdifferential of $V'_{t-1,1}(\alpha_{t-1,1})$ is a bounded subset of $\mathbb{R}^+ \setminus \{0\}$.

A.4. $V_{t-1,1}(\alpha_{t-1,1})$ is strictly convex.

A.5. $0 = V'_{t-1,1}(0) < V'_{t-1,1}(1) < 1$

A.6. $p'\left(V'_{t-1,1}(1)\right) \cdot \left(V_{t-1,1}(1) - V'_{t-1,1}(1)\right) + 1 - p\left(V'_{t-1,1}(1)\right) < 1$

S.2 Proof of Lemma 1 and Proposition 1

Remark 2 In every expression involving the second degree differential V'' of the value function either explicitly or implicitly in the expression $\alpha'_{t-1,1}$ or $r'_{t,1}$, V'' denotes an arbitrary element of the subdifferential of V' as the value function is not two times differentiable in a finite number of points. In the following we only require the subdifferential of V' to be a bounded subset of $\mathbb{R}^+ \setminus \{0\}$. This property is shown in the proof of A.3. Moreover, for $\alpha \in \{0,1\}$ all differentials are to be understood as one-sided limits of difference quotients.

Regarding $\alpha_{t,1} > 0$, we start our analysis by restricting the set of feasible prices w.l.o.g. (Lemma 1): For $r_{t,1} \in \{0,1\}, V_{t,1}(\alpha_{t,1}) \leq V_{t-1,1}(\alpha_{t-1,1})$, and, therefore, $r_{t,1} \in \{0,1\}$ is not optimal.

(12) yields $\alpha_{t-1,0} = \frac{\alpha_{t,1} - \left(1 - p(r_{t,1})\right) \cdot \alpha_{t-1,1}}{p(r_{t,1})}$ since $p(r_{t,1}) \in (0,1)\ \forall r_{t,1} \in (0,1)$ (cf. P.2 and P.4). This leads to:

$$V_{t,1}(\alpha_{t,1}) = \max_{r_{t,1}} \min_{\alpha_{t-1,1}} F(r_{t,1}, \alpha_{t-1,1})$$

$$\text{with } F(r_{t,1}, \alpha_{t-1,1}) = \left(1 - p(r_{t,1})\right) \cdot \left(V_{t-1,1}(\alpha_{t-1,1}) - r_{t,1} \cdot \alpha_{t-1,1}\right) + r_{t,1} \cdot \alpha_{t,1}. \quad (24)$$

$$\text{s.t. } \frac{\alpha_{t,1} - \left(1 - p(r_{t,1})\right) \cdot \alpha_{t-1,1}}{p(r_{t,1})} \in [0,1] \quad (25)$$

$$\alpha_{t-1,1} \in [0,1] \quad (26)$$

$$r_{t,1} \in (0,1) \quad (27)$$

We first take a look at the inner minimization and calculate:

$$\frac{d}{d\alpha_{t-1,1}} F(r_{t,1}, \alpha_{t-1,1}) = \left(1 - p(r_{t,1})\right) \cdot \left(V'_{t-1,1}(\alpha_{t-1,1}) - r_{t,1}\right)$$

$$\frac{d^2}{d\alpha_{t-1,1}^2} F(r_{t,1}, \alpha_{t-1,1}) = \left(1 - p(r_{t,1})\right) \cdot V''_{t-1,1}(\alpha_{t-1,1}) > 0 \text{ (cf. A.3, P.2, P.4 and Lemma 1)}$$

We now distinguish two (overlapping) cases with respect to $r_{t,1}$.

Case 1: $r_{t,1} \in \left(0, V'_{t-1,1}(1)\right]$. $\alpha_{t-1,1}(r_{t,1}) = \left(V'_{t-1,1}\right)^{-1}(r_{t,1})$ is the global minimum of $F(r_{t,1}, \alpha_{t-1,1})$.

Case 2: $r_{t,1} \in \left[V'_{t-1,1}(1), 1\right)$. Since $\frac{d}{d\alpha_{t-1,1}} F(r_{t,1}, \alpha_{t-1,1}) \leq 0$, $\alpha_{t-1,1}(r_{t,1})$ is set to the maximal feasible value.

We now solve the outer maximization problem:

$$\frac{d}{d\, r_{t,1}} F\left(r_{t,1}, \alpha_{t-1,1}(r_{t,1})\right) = -p'(r_{t,1}) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}(r_{t,1})\right) - r_{t,1} \cdot \alpha_{t-1,1}(r_{t,1})\right) - \left(1 - p(r_{t,1})\right) \cdot \alpha_{t-1,1}(r_{t,1}) + \alpha_{t,1}$$

$$\frac{d^2}{d\, r_{t,1}^2} F\left(r_{t,1}, \alpha_{t-1,1}(r_{t,1})\right) = -p''(r_{t,1}) \cdot \underbrace{\left(V_{t-1,1}\left(\alpha_{t-1,1}(r_{t,1})\right) - r_{t,1} \cdot \alpha_{t-1,1}(r_{t,1})\right)}_{<0,\text{ A.4}} + 2\underbrace{p'(r_{t,1})}_{<0,\text{ P.2}} \cdot \underbrace{\alpha_{t-1,1}(r_{t,1})}_{>0} - \underbrace{\left(1 - p(r_{t,1})\right)}_{>0} \cdot \underbrace{\alpha'_{t-1,1}(r_{t,1})}_{\geq 0,\text{ Remark 2}} = -p''(r_{t,1}) \cdot \underbrace{V_{t-1,1}\left(\alpha_{t-1,1}(r_{t,1})\right)}_{>0} + \underbrace{\alpha_{t-1,1}(r_{t,1})}_{>0} \cdot \underbrace{\left(p''(r_{t,1}) \cdot r_{t,1} + 2p'(r_{t,1})\right)}_{<0,\text{ P.3}} - \underbrace{\left(1 - p(r_{t,1})\right)}_{>0} \cdot \underbrace{\alpha'_{t-1,1}(r_{t,1})}_{\geq 0,\text{ Remark 2}} < 0$$

To show that $\frac{d^2}{d\, r_{t,1}^2} F\left(r_{t,1}, \alpha_{t-1,1}(r_{t,1})\right) < 0$ is independent of the sign of $p''(r_{t,1})$, we resort the expression in the last equality. Now, we can respectively observe the inequality for $p''(r_{t,1}) \leq 0$ and $p''(r_{t,1}) > 0$ with the left and right hand side of the last equality. Thus, $F\left(r_{t,1}, \alpha_{t-1,1}(r_{t,1})\right)$ is strictly concave in $r_{t,1}$. There is at most one inner maximum such that $\frac{d}{d\, r_{t,1}} F\left(r_{t,1}(\alpha_{t,1}), \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) = 0$, where $r_{t,1}(\alpha_{t,1})$ is the unique solution to the equation

$$\alpha_{t,1} = p'\left(r_{t,1}(\alpha_{t,1})\right) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) + \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right) \quad (28)$$

Now we define $\alpha_{t,1}^{PI_{t-1}} = p'\left(V'_{t-1,1}(1)\right) \cdot \left(V_{t-1,1}(1) - V'_{t-1,1}(1)\right) + 1 - p\left(V'_{t-1,1}(1)\right)$ and we split the optimization in four cases, based on the values of $\alpha_{t,1}$ and $r_{t,1}$ (Table S.1).

Table S.1: Cases considered in the optimization

	$\boldsymbol{r_{t,1} \in \left(0, V'_{t-1,1}(1)\right]}$	$\boldsymbol{r_{t,1} \in \left[V'_{t-1,1}(1), 1\right)}$
$\boldsymbol{\alpha_{t,1} \in (0, \alpha_{t,1}^{PI_{t-1}}]}$	Case 1a	Case 2a
$\boldsymbol{\alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1]}$	Case 1b	Case 2b

In these cases, constraint (27) is satisfied by definition.

Case 1a: $\alpha_{t-1,1}(r_{t,1}) = \left(V'_{t-1,1}\right)^{-1}(r_{t,1})$ and $r_{t,1}$ is given implicitly by equation (28). Since $\frac{d}{d\,r_{t,1}} F\left(r_{t,1}, \alpha_{t-1,1}(r_{t,1})\right)$ is continuous, $\frac{d}{d\,r_{t,1}} F\left(0, \alpha_{t-1,1}(0)\right) > 0$ and $\frac{d}{d\,r_{t,1}} F\left(V'_{t-1,1}(1), \alpha_{t-1,1}\left(V'_{t-1,1}(1)\right)\right) \leq 0$, the existence of a solution $r_{t,1}(\alpha_{t,1})$ to equation (28) is guaranteed by the intermediate value theorem.

Constraint (26) holds because $\alpha_{t-1,1} > 0, \frac{d}{d\,r_{t,1}} \alpha_{t-1,1}(r_{t,1}) > 0$ and $\alpha_{t-1,1}\left(V'_{t-1,1}(1)\right) = 1$. Constraint (25) remains to be shown. We have

$$\frac{\alpha_{t,1} - \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \left(V'_{t-1,1}\right)^{-1}\left(r_{t,1}(\alpha_{t,1})\right)}{p\left(r_{t,1}(\alpha_{t,1})\right)} \geq 0$$

$$\Leftrightarrow \underbrace{p'\left(r_{t,1}(\alpha_{t,1})\right)}_{<0} \cdot \underbrace{\left(V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - r_{t,1}(\alpha_{t,1}) \cdot \left(V'_{t-1,1}\right)^{-1}\left(r_{t,1}(\alpha_{t,1})\right)\right)}_{<0} \geq 0$$

In addition to that, we have

$$\frac{\alpha_{t,1} - \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \left(V'_{t-1,1}\right)^{-1}\left(r_{t,1}(\alpha_{t,1})\right)}{p\left(r_{t,1}(\alpha_{t,1})\right)} \leq 1$$

$$\Leftrightarrow 0 \leq p\left(r_{t,1}(\alpha_{t,1})\right) - p'\left(r_{t,1}(\alpha_{t,1})\right) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - r_{t,1}(\alpha_{t,1}) \cdot \left(V'_{t-1,1}\right)^{-1}\left(r_{t,1}(\alpha_{t,1})\right)\right)$$

Since the right hand side of the inequality is decreasing in $r_{t,1}(\alpha_{t,1})$, the inequality holds due to A.6.

Case 1b: Since $\frac{d}{d\,r_{t,1}} F\big(r_{t,1}, \alpha_{t-1,1}\big) \geq 0$ for all $\alpha_{t-1,1}$, we choose $r_{t,1}\big(\alpha_{t,1}\big) = V'_{t-1,1}(1)$. Showing constraint (26) holds is identical to case 1a. We further have $\frac{\alpha_{t,1} - \left(1 - p\left(V'_{t-1,1}(1)\right)\right)}{p\left(V'_{t-1,1}(1)\right)} \geq 0$, as $\alpha_{t,1} > \alpha_{t,1}^{PI_{t-1}}$ and $\frac{\alpha_{t,1} - \left(1 - p\left(V'_{t-1,1}(1)\right)\right)}{p\left(V'_{t-1,1}(1)\right)} \leq 1$ obviously holds.

Case 2a: In this case, the choice of $\alpha_{t-1,1}\big(r_{t,1}\big) = 1$ might violate constraint (25) if $\alpha_{t,1} < 1 - p\big(r_{t,1}\big)$. By setting $\alpha_{t-1,1}\big(r_{t,1}\big) = \min\{\frac{\alpha_{t,1}}{1 - p(r_{t,1})}, 1\}$, the inner minimization problem is solved und constraints (25) and (26) hold. Since $\frac{d}{d\,r_{t,1}} F\left(r_{t,1}, \alpha_{t-1,1}\big(r_{t,1}\big)\right) \leq 0$, $r_{t,1}(\alpha_{t,1}) = V'_{t-1,1}(1)$.

Case 2b: In this case, we have $\alpha_{t-1,1}\big(r_{t,1}\big) = 1$. However, $r_{t,1}\big(\alpha_{t,1}\big)$ is given implicitly by (28). Since $\frac{d}{d\,r_{t,1}} F\big(r_{t,1}, 1\big)$ is continuous, $\frac{d}{d\,r_{t,1}} F\big(V'_{t-1,1}(1), 1\big) > 0$ and $\frac{d}{d\,r_{t,1}} F(1,1) \leq 0$, the existence of a solution to equation (28) is guaranteed by the intermediate value theorem. Constraint (26) is satisfied. Regarding constraint (25), $\frac{\alpha_{t,1} - \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right)}{p\left(r_{t,1}(\alpha_{t,1})\right)} \leq 1$ obviously holds. Moreover, $\frac{\alpha_{t,1} - \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right)}{p\left(r_{t,1}(\alpha_{t,1})\right)} \geq 0 \underbrace{\Leftrightarrow}_{(28)} \underbrace{p'\left(r_{t,1}(\alpha_{t,1})\right)}_{<0} \cdot \underbrace{\left(V_{t-1,1}(1) - r_{t,1}(\alpha_{t,1})\right)}_{<0,\ \text{A.4}} \geq 0$

Summary: Now, note that $r_{t,1}(\alpha_{t,1}) = V'_{t-1,1}(1)$ determined in case 2a is also considered in case 1a. Thus, case 1a dominates case 2a. Similarly, case 2b dominates case 1b. To sum up, for $\alpha_{t,1} = 0$, we have $\alpha_{t-1,1}\left(r_{t,1}(0)\right) = 0$ and $r_{t,1}(0) = 0$. For $\alpha_{t,1} \in (0, \alpha_{t,1}^{PI_{t-1}}]$, we obtain $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right) = \left(V'_{t-1,1}\right)^{-1}\left(r_{t,1}(\alpha_{t,1})\right)$ and for $\alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1]$, we obtain $\alpha_{t-1,1}\big(r_{t,1}\big) = 1$. In both cases $r_{t,1}\big(\alpha_{t,1}\big)$ is given by (28). Substituting this into (11), we obtain $V_{t,1}\big(\alpha_{t,1}\big)$.

S.3 Proof of Proposition 2

We start with the first section, i.e. $\alpha_{t,1} \in [0, \alpha_{t,1}^{PI_{t-1}}]$. The optimal price $r_{t,1}\big(\alpha_{t,1}\big)$ is given implicitly by Proposition 1.

Remark 3 $r_{t,1}\big(\alpha_{t,1}\big)$ is continuously differentiable except at points where the value function is not two times differentiable (see Remark 2). This follows from the implicit function theorem together with

$$\frac{d}{d\,r_{t,1}(\alpha_{t,1})}\Big[\alpha_{t,1} - p'\left(r_{t,1}(\alpha_{t,1})\right) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\Big] < 0.$$

$$1 \underbrace{=}_{(15)} \frac{d}{d\,\alpha_{t,1}}\Big[p'\left(r_{t,1}(\alpha_{t,1})\right) \cdot \left(V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) + \left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\Big]$$

$$\underset{(14)}{=} r'_{t,1}(\alpha_{t,1}) \cdot \left\{ p''\left(r_{t,1}(\alpha_{t,1})\right) \cdot \left(\underbrace{V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) - r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)}_{<0,\ \text{A.4 and (14)}} \right) - \right.$$

$$\left. 2 \underbrace{\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)}_{\geq 0} \cdot \underbrace{p'\left(r_{t,1}(\alpha_{t,1})\right)}_{<0,\ \text{P.2}} + \underbrace{\left(1 - p\left(r_{t,1}(\alpha_{t,1})\right)\right)}_{>0} \cdot \underbrace{\alpha'_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)}_{>0,\ \text{Remark 2}} \right\}$$

As the part in curly brackets is positive (cf. the argumentation of $\frac{d^2}{d\, r_{t,1}^2} F\left(r_{t,1}, \alpha_{t-1,1}(r_{t,1})\right) < 0$ in Section S.2), we can see that $r'_{t,1}(\alpha_{t,1})$ has to be positive as well.

The second section, i.e. $\alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1]$, can analogously be shown and is omitted.

As $r'_{t,1}(\alpha_{t,1})$ is positive on every section, $r_{t,1}(\alpha_{t,1})$ is strictly monotonically increasing in $\alpha_{t,1}$.

S.4 Proof of Proposition 3

We have to check the continuity in the cases $\alpha_{t,1} \in [0, \alpha_{t,1}^{PI_{t-1}})$, $\alpha_{t,1} = \alpha_{t,1}^{PI_{t-1}}$ and $\alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1]$:

The proof for the first case is predominantly algebra. It holds:

$$\forall \hat{\alpha}_{t,1} \in \left(0, \alpha_{t,1}^{PI_{t-1}}\right): \lim_{\underline{\alpha}_{t,1} \nearrow \hat{\alpha}_{t,1}} r_{t,1}(\underline{\alpha}_{t,1}) = r_{t,1}(\hat{\alpha}_{t,1}) = \lim_{\overline{\alpha}_{t,1} \searrow \hat{\alpha}_{t,1}} r_{t,1}\left(\overline{\alpha}_{t,1}\right)$$

Considering Proposition 2 and the bound $\hat{\alpha}_{t,1}$ of the sequences, clearly, both limits exist. The equality can be shown by taking the one-sided limits of both sides of (15) and using P.1, A.2 and the uniqueness of the solution of (15). Moreover, $\forall \epsilon > 0\ \exists \delta > 0$: $r_{t,1}(\alpha_{t,1}) \in (0, \epsilon)\ \forall \alpha_{t,1} \in (0, \delta)$ and, thus, $\lim_{\alpha_{t,1} \searrow 0} r_{t,1}(\alpha_{t,1}) = 0 = r_{t,1}(0)$.

The proof for the second and third case is analogous to the first one.

S.5 Proof of Lemma 2

Lemma 2 *The optimal price $r_{t,1}(\alpha_{t,1})$ at $\alpha_{t,1}$ equals the slope of the value function at this point, i.e. $r_{t,1}(\alpha_{t,1}) = V'_{t,1}(\alpha_{t,1})$.*

We start with the differential of the objective function in the first section, i.e. $\alpha_{t,1} \in [0, \alpha_{t,1}^{PI_{t-1}}]$.

$$\frac{d}{d\,\alpha_{t,1}} V_{t,1}(\alpha_{t,1}) =$$

$$r'_{t,1}(\alpha_{t,1}) \cdot \Big\{\alpha_{t,1} - p'\big(r_{t,1}(\alpha_{t,1})\big) \cdot \Big(V_{t-1,1}\Big(\alpha_{t-1,1}\big(r_{t,1}(\alpha_{t,1})\big)\Big) - r_{t,1}(\alpha_{t,1}) \cdot \alpha_{t-1,1}\big(r_{t,1}(\alpha_{t,1})\big)\Big) -$$

$$\Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big) \cdot \alpha_{t-1,1}\big(r_{t,1}(\alpha_{t,1})\big)\Big\} + \Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big) \cdot \alpha'_{t-1,1}\big(r_{t,1}(\alpha_{t,1})\big) \cdot r'_{t,1}(\alpha_{t,1}) \cdot$$

$$\Big[V'_{t-1,1}\Big(\alpha_{t-1,1}\big(r_{t,1}(\alpha_{t,1})\big)\Big) - r_{t,1}(\alpha_{t,1})\Big] + r_{t,1}(\alpha_{t,1}) \underbrace{=}_{\text{Proposition 1}} r_{t,1}(\alpha_{t,1})$$

We continue with the differential of the objective function in the second section, i.e. $\alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1]$.

$$\frac{d}{d\,\alpha_{t,1}} V_{t,1}(\alpha_{t,1}) = r_{t,1}(\alpha_{t,1}) + r'_{t,1}(\alpha_{t,1}) \cdot$$

$$\underbrace{\Big[\alpha_{t,1} - p'\big(r_{t,1}(\alpha_{t,1})\big) \cdot \big(V_{t-1,1}(1) - r_{t,1}(\alpha_{t,1})\big) - \Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big)\Big]}_{=0,\ \text{Proposition 1}} = r_{t,1}(\alpha_{t,1})$$

Together with the continuity of $r_{t,1}(\alpha_{t,1})$ at $\alpha_{t,1}^{PI_{t-1}}$, the proposition is proven.

S.6 Proof of Proposition 4

Proposition 4 can be shown by combining Proposition 1 and Lemma 2.

S.7 Proof of Proposition 5

We consider two cases. Firstly, we assume $\alpha_{t,1} = 1$. Secondly, we assume $\alpha_{t,1} < 1$.

Case 1: As $\alpha_{t,1} = 1$, the Value-at-Risk at this level corresponds to the highest possible optimal price with a positive probability of occurrence, i.e. $VaR_1 \underbrace{=}_{\text{Proposition 4}} \max\{r_{t,1}(1), \dots, r_{1,1}(1)\} = r_{t,1}(1)$.

Case 2: $\alpha_{t,1} < 1 \Rightarrow \exists n \leq t: \alpha_{t-n,1} = 1 \wedge \alpha_{t-n+1,1} < 1$. Thus, $\alpha_{t,1} > \Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big) \cdot \alpha_{t-1,1} > \dots > \Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big)^n \cdot \alpha_{t-n,1} = \Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big)^n$ and $r_{t,1}(\alpha_{t,1}) \geq r_{t-i,1}(\alpha_{t-i,1})\ \forall i \leq t$ (cf. (11) - (13) and Proposition 4). The probability to sell the item for the highest price $r_{t,1}(\alpha_{t,1})$ is given by $1 - \Big(1 - p\big(r_{t,1}(\alpha_{t,1})\big)\Big)^n$. As $\alpha_{t,1}$ exceeds the probability of not selling at the highest price $r_{t,1}(\alpha_{t,1})$, the Value-at-Risk at $\alpha_{t,1}$ is $r_{t,1}(\alpha_{t,1})$.

S.8 Proof of Proposition 6

In this section, we show the proposition by induction. To simplify the notation, we write $\alpha^*_{t-1,1}$ and $\alpha^*_{t-2,1}$ instead of $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$ and $\alpha_{t-2,1}\left(r_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right)\right)$, respectively, and $r^*_{t,1}$ and $r^*_{t-1,1}$ instead of $r_{t,1}(\alpha_{t,1})$ and $r_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right)$, respectively.

We omit the trivial induction basis and check the following cases in the induction step: $\alpha_{t,1} = 0$, $\alpha_{t,1} \in \left(0, \alpha_{t,1}^{PI_{t-1}}\right]$ and $\alpha_{t,1} \in (\alpha_{t,1}^{PI_{t-1}}, 1]$.

The first and the third case are respectively covered by Remark 1 and Proposition 1.

For the second case, we use Proposition 1, Proposition 4 and the induction hypothesis $\alpha^*_{t-1,1} \leq \alpha^*_{t-2,1}$.

$$\alpha_{t,1} = p'\left(r^*_{t-1,1}\right) \cdot \left(V_{t-1,1}\left(\alpha^*_{t-1,1}\right) - r^*_{t-1,1} \cdot \alpha^*_{t-1,1}\right) + \left(1 - p\left(r^*_{t-1,1}\right)\right) \cdot \alpha^*_{t-1,1} = p'\left(r^*_{t-1,1}\right) \cdot$$
$$\left(1 - p\left(r^*_{t-1,1}\right)\right) \cdot \left(V_{t-2,1}\left(\alpha^*_{t-2,1}\right) - r^*_{t-1,1} \cdot \alpha^*_{t-2,1}\right) + \left(1 - p\left(r^*_{t-1,1}\right)\right) \cdot \alpha^*_{t-1,1}$$
$$\leq p'\left(r^*_{t-1,1}\right) \cdot \left(V_{t-2,1}\left(\alpha^*_{t-2,1}\right) - r^*_{t-1,1} \cdot \alpha^*_{t-2,1}\right) + \left(1 - p\left(r^*_{t-1,1}\right)\right) \cdot \alpha^*_{t-2,1} = \alpha^*_{t-1,1}$$

The inequality follows by substituting the induction hypothesis together with P.2 and A.4.

S.9 Proof of the properties of the value function

In this section, we show Properties A.1–A.6 by induction. This requires first explicitly solving the optimization problem inherent in $V_{1,1}\left(\alpha_{1,1}\right)$ and showing that A.1–A.6 hold for $t - 1 = 1$ (induction basis). Both is rather straightforward and omitted to save space. In the induction step, we assume that the properties hold for $t - 1$ (induction hypothesis, IH) and, thus, are able to use the results of Proposition 1–Proposition 4 to show that the Properties A.1–A.6 also hold for t as well:

As $V_{t,1}(0) = 0$ (cf. Remark 1) and $V_{t,1}(1) = \left(1 - p\left(r_{t,1}(1)\right)\right) \cdot V_{t-1,1}(1) + p\left(r_{t,1}(1)\right) \cdot r_{t,1}(1) \in (0,1)$ (cf. Proposition 1, Lemma 1, P.4 and IH), Property A.1 holds. To show Properties A.2–A.6 we use Lemma 2, i.e. $V'_{t,1}\left(\alpha_{t,1}\right) = r_{t,1}\left(\alpha_{t,1}\right)$. Property A.2 and A.5 follow by additionally using Lemma 1 and Proposition 3, and Property A.3 by Remark 3. Property A.4 holds because $V_{t,1}\left(\alpha_{t,1}\right)$ is continuously differentiable on the whole domain and $V''_{t,1}\left(\alpha_{t,1}\right) > 0$, thus, we know that $V_{t,1}\left(\alpha_{t,1}\right)$ is strictly convex for every $\alpha_{t,1} \in (0,1)$. To show Property A.6, we observe that

$$p'\left(V'_{t,1}(1)\right)\cdot\left(V_{t,1}(1)-V'_{t,1}(1)\right)+1-p\left(V'_{t,1}(1)\right)=$$

$$\underbrace{p'\left(r_{t,1}(1)\right)\cdot\left(V_{t-1,1}(1)-r_{t,1}(1)\right)+1-p\left(r_{t,1}(1)\right)}_{=1,\ (15)\text{ with }\alpha_{t,1}=1}+p'\left(r_{t,1}(1)\right)\cdot\left(V_{t,1}(1)-V_{t-1,1}(1)\right)<1$$

Where the final inequality follows from P.2 and the well-known strict monotonicity of $V_{t,1}(1)$ in time.

S.10 Proof of Proposition 7

S.10.1 Proof of (19)

We show (19) by induction over t. Note that the value function (11) simplifies to

$$V_{t,1}(1)=\max_{r_{t,1}}\left\{\left(1-r_{t,1}\right)\cdot r_{t,1}+r_{t,1}\cdot V_{t-1,1}(1)\right\}\text{ with }V_{0,1}(1)=0 \tag{29}$$

The induction hypothesis (19) obviously holds for $t=1$. The inductive step from $t-1$ to t follows from (29) and Proposition 1. Please note that Proposition 1 considerably simplifies as we only consider the special case of uniformly distributed WTPs, i.e. $p\left(r_{t,1}\right)=1-r_{t,1}$ in Proposition 7.

S.10.2 Proof of (20), (21), (22) and (23) – base case

With $V_{0,1}=0$, (17) and Proposition 1, we have that (20), (21) and (22) hold for $t=1$ and $j=1$.

Moreover, (23) holds for $t=2$ (cf. Lemma 1, Proposition 1 and A.5).

S.10.3 Proof of (20), (21), (22) and (23) – inductive step

Now, suppose (20), (21), (22) and (23) hold in time period $t-1$.

Due to the strict monotonicity and continuity of $r_{t,1}\left(\alpha_{t,1}\right)$ and $\alpha_{t-1,1}\left(r_{t,1}\right)$, (23) is equivalent to:

$$\alpha_{t,1}=\alpha_{t,1}^{PI_j}\text{ solves }\alpha_{t-1,1}^{PI_{\min\{j,t-1\}}}=\alpha_{t-1,1}\left(r_{t,1}\left(\alpha_{t,1}\right)\right) \tag{30}$$

Case $j=t$: For $j=t$, i.e. $\alpha_{t,1}\in S_{t,1}^{t}$, and, therefore, $\alpha_{t,1}>\alpha_{t,1}^{PI_{t-1}}=2V'_{t-1,1}(1)-V_{t-1,1}(1)$. The equality follows from Proposition 1 and Lemma 2. Thus, equation (21) directly follows from Proposition 1. Using this and algebra with $V_{t,1}\left(\alpha_{t,1}\right)=r_{t,1}^{2}$, (20) can be shown and (22) is trivial. Moreover, it holds that $\alpha_{t-1,1}\left(r_{t,1}\left(\alpha_{t,1}\right)\right)=1$. Thus, $\alpha_{t-1,1}\in S_{t-1,1}^{t-1}$ for $\alpha_{t,1}\in S_{t,1}^{t}$ and equation (23) holds.

Case $j < t$: For $j < t$, $\alpha_{t,1} \le \alpha_{t,1}^{PI_{t-1}} = 2V'_{t-1,1}(1) - V_{t-1,1}(1)$. Substituting $t-1$ into (20) and (21) yields the following formulation of the induction hypothesis:

$$V_{t-1,1}(\alpha_{t-1,1}) = -r_{t-1,1}^{t-j+1} + r_{t-1,1}^{t-j} \cdot V_{j-1,1}(1) + r_{t-1,1} \cdot \alpha_{t-1,1} \tag{31}$$

$$-(t-j+1) \cdot r_{t-1,1}^{t-j} + (t-j) \cdot r_{t-1,1}^{t-j-1} \cdot V_{j-1,1}(1) + \alpha_{t-1,1} = 0 \tag{32}$$

We substitute the induction hypothesis (32) for $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$ in (15). As $\alpha_{t,1} \le \alpha_{t,1}^{PI_{t-1}}$, we also substitute $r_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) = r_{t,1}(\alpha_{t,1})$ (cf. Proposition 4) and obtain:

$$0 = V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) + \alpha_{t,1} - 2\left((t-j+1) \cdot r_{t,1}^{t-j+1}(\alpha_{t,1}) - (t-j) \cdot r_{t,1}^{t-j}(\alpha_{t,1}) \cdot V_{j-1,1}(1)\right) \tag{33}$$

Next, we consider (31) from the induction hypothesis and substitute $\alpha_{t-1,1}$ using (32) and again $r_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) = r_{t,1}(\alpha_{t,1})$ to obtain:

$$V_{t-1,1}(\alpha_{t-1,1}) = (1-t+j) \cdot r_{t,1}^{t-j} \cdot V_{j-1,1}(1) + (t-j) \cdot r_{t,1}^{t-j+1}(\alpha_{t,1}) \tag{34}$$

Now, part (21) of the induction hypothesis is shown by substituting (34) into (33).

(20) is shown using $V_{t,1}(\alpha_{t,1})$ from Proposition 1 and substituting $\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)$ using (32) and $V_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right)$ using (31) as well as $r_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) = r_{t,1}(\alpha_{t,1})$ (cf. Proposition 4).

To show (30), and, thereby, (23), remember that $V'_{t-1,1}\left(\alpha_{t-1,1}\left(r_{t,1}(\alpha_{t,1})\right)\right) = r_{t,1}(\alpha_{t,1})$ according to Proposition 1, as $\alpha_{t,1} \le \alpha_{t,1}^{PI_{t-1}}$. By the induction hypothesis, we know that $r_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right) = r^{PI_j}, \forall j \le t-1$. Choose the lowest $\hat{\alpha}_{t,1}$, so that $\alpha_{t-1,1}^{PI_j} = \alpha_{t-1,1}\left(r_{t,1}(\hat{\alpha}_{t,1})\right)$. If $\alpha_{t-1,1}^{PI_j} = 1$, then $\hat{\alpha}_{t,1} = \alpha_{t,1}^{PI_{t-1}}$ (cf. Proposition 1). If $\alpha_{t-1,1}^{PI_j} < 1$, then $\hat{\alpha}_{t,1} < \alpha_{t,1}^{PI_{t-1}}$. Summing up, $\hat{\alpha}_{t,1} \le \alpha_{t,1}^{PI_{t-1}}$ holds, and we have $r_{t,1}(\hat{\alpha}_{t,1}) = r_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right) = r^{PI_j}$ where the first equality follows Proposition 4.

Now we have to show that $\hat{\alpha}_{t,1} = \alpha_{t,1}^{PI_j}$ and, thereby, (22) and (23) hold. From Proposition 1 we use the definition of $r_{t,1}(\alpha_{t,1})$ to calculate $\hat{\alpha}_{t,1}$ as well as $V_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right) = \left(r_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right)\right)^2 \cdot \alpha_{t-2,1}\left(r_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right)\right)$ from the equation for $V_{t,1}(\alpha_{t,1})$ to obtain:

$$\hat{\alpha}_{t,1} = 2\underbrace{r_{t,1}(\hat{\alpha}_{t,1})}_{=r^{PI_j}} \cdot \alpha_{t-1,1}^{PI_j} - \Bigg(\underbrace{r_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right)}_{=r^{PI_j}}\Bigg)^2 \cdot \underbrace{\alpha_{t-2,1}\left(r_{t-1,1}\left(\alpha_{t-1,1}^{PI_j}\right)\right)}_{=\alpha_{t-2,1}^{PI_j}\text{(IH)}} \underbrace{=}_{(18)} \alpha_{t,1}^{PI_j}$$

As $r_{t,1}(\alpha_{t,1})$ in Proposition 1 is unique, $\hat{\alpha}_{t,1}$ is also the smallest solution. This completes the inductive step for (30), and, thereby (22) and (23).

S.11 Comparison of mechanism A and an exponential utility function

Besides the three benchmark approaches *EV-Dyn*, *CVaR-Fix*, and *EV-Fix*, we also considered a popular and easy to calculate approach from literature. The exponential utility function with constant absolute risk aversion (CARA, $-e^{-\gamma R}$) with parameter γ can be easily integrated into a dynamic program (see e.g. Lim and Shanthikumar (2007)) without increasing the computational burden. To determine γ, we tested a broad range of γ-values to find the one with the highest $CVaR_\alpha$ for a given α. We repeated this procedure for several states (c, t). Figure S.1 illustrates our analyses for $CVaR_{0.3}$ and two states: $T = 10$ and $C = 1$ (left part) and $C = 5$ (right part). We observe that $CVaR_{0.3}$ is highest at $\gamma = 3.75$ for $C = 1$ and $\gamma = 2.5$ for $C = 5$. Thus, already this small example shows that there is no simple matching of optimal γ to α, but the γ also depends on the state. This makes it impossible to choose the best γ a priori. Only a simulation based optimization analogous to Koch et al. (2016) is possible. However, as expected, CVaR obtained when optimizing the exponential utility is considerably lower compared to the optimization of CVaR with $A_{0.0001}$ (see also

Table S.2) and the difference decreases in α.

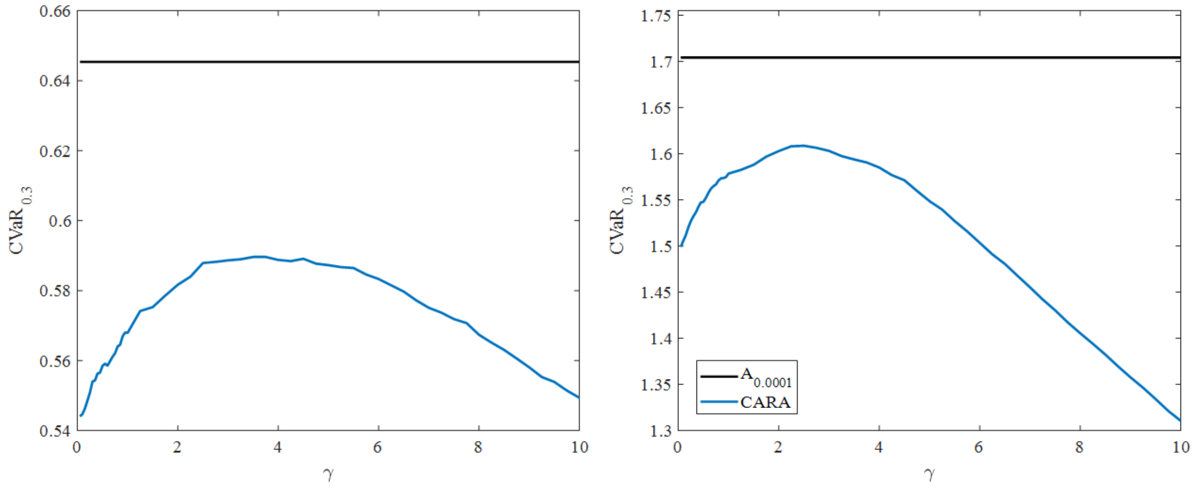

Figure S.1: $CVaR_\alpha$ obtained when optimizing exponential utility (CARA) and $A_{0.0001}$ for α =0.3, T=10 with C=1 (left) and C=5 (right)

Table S.2: CVaR obtained from an exponential utility-optimal policy with the best value for γ relative to $A_{0.0001}$

T=10	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.7$
C=1	80.18%	82.90%	91.36%	98.62%
C=5	92.86%	92.72%	94.40%	98.10%
C=10	96.08%	96.53%	97.57%	99.15%